

# Duality Relation for the Hilbert Series of Almost Symmetric Numerical Semigroups

Leonid G. Fel

Department of Civil Engineering, Technion, Haifa 3200, Israel

*e-mail: lfel@tx.technion.ac.il*

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## Abstract

We derive the duality relation for the Hilbert series  $H(\mathbf{d}^m; z)$  of almost symmetric numerical semigroup  $S(\mathbf{d}^m)$  combining it with its dual  $H(\mathbf{d}^m; z^{-1})$ . On this basis we establish the bijection between the multiset of degrees of the syzygy terms and the multiset of the gaps  $F_j$ , generators  $d_i$  and their linear combinations. We present the relations for the sums of the Betti numbers of even and odd indices separately. We apply the duality relation to the simple case of the almost symmetric semigroups of maximal embedding dimension, and give the necessary and efficient conditions for minimal set  $\mathbf{d}^m$  to generate such semigroups.

**Keywords:** Almost symmetric semigroups, the Hilbert series, the Betti numbers.

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# 1 Introduction

This article deals mainly with almost symmetric numerical semigroups which were introduced in [1] and present a special class of nonsymmetric numerical semigroups  $S(\mathbf{d}^m)$  in  $\mathbb{N} \cup \{0\}$ . Throughout the article we assume that  $S(\mathbf{d}^m)$  is finitely generated by a minimal set of positive integers  $\mathbf{d}^m = \{d_1, \dots, d_m\}$  with finite complement in  $\mathbb{N}$ ,  $\#\{\mathbb{N} \setminus S(\mathbf{d}^m)\} < \infty$ . We study the generating function  $H(\mathbf{d}^m; z)$  of such semigroup  $S(\mathbf{d}^m)$ ,

$$H(\mathbf{d}^m; z) = \sum_{s \in S(\mathbf{d}^m)} z^s, \quad (1)$$

which is referred to as *the Hilbert series* of  $S(\mathbf{d}^m)$ .

Recall the main definitions and known facts on numerical semigroups which are necessary here. A semigroup  $S(\mathbf{d}^m) = \{s \in \mathbb{N} \cup \{0\} \mid s = \sum_{i=1}^m x_i d_i, x_i \in \mathbb{N} \cup \{0\}\}$ , is said to be generated by *minimal set* of  $m$  natural numbers  $d_1 < \dots < d_m$ ,  $\gcd(d_1, \dots, d_m) = 1$ , if neither of its elements is linearly representable by the rest of elements. It is classically known that  $d_1 \geq m$  [8] where  $d_1$  and  $m$  are called *the multiplicity* and *the embedding dimension (edim)* of the semigroup, respectively. If equality  $d_1 = m$  holds then the semigroup  $S(\mathbf{d}^m)$  is called of maximal edim. *The conductor*  $c(\mathbf{d}^m)$  of semigroup  $S(\mathbf{d}^m)$  is defined by  $c(\mathbf{d}^m) := \min\{s \in S(\mathbf{d}^m) \mid s + \mathbb{N} \cup \{0\} \subset S(\mathbf{d}^m)\}$  and related to *the Frobenius number* of semigroup,  $F(\mathbf{d}^m) = c(\mathbf{d}^m) - 1$ .

Denote by  $\Delta(\mathbf{d}^m)$  the complement of  $S(\mathbf{d}^m)$  in  $\mathbb{N}$ , i.e.  $\Delta(\mathbf{d}^m) = \mathbb{N} \setminus S(\mathbf{d}^m)$ , and call it the set of gaps. The cardinality ( $\#$ ) of  $\Delta(\mathbf{d}^m)$  is called *the genus* of  $S(\mathbf{d}^m)$ ,  $G(\mathbf{d}^m) := \#\Delta(\mathbf{d}^m)$ . For the set  $\Delta(\mathbf{d}^m)$  introduce the generating function  $\Phi(\mathbf{d}^m; z)$  which is related to the Hilbert series,

$$\Phi(\mathbf{d}^m; z) = \sum_{s \in \Delta(\mathbf{d}^m)} z^s, \quad \Phi(\mathbf{d}^m; z) + H(\mathbf{d}^m; z) = \frac{1}{1-z}.$$

Denote by  $t(\mathbf{d}^m)$  *the type* of the numerical semigroup  $S(\mathbf{d}^m)$  which coincides with cardinality of set  $S'(\mathbf{d}^m)$  that is defined [8] as follows,

$$S'(\mathbf{d}^m) = \{F_j \in \mathbb{Z} \mid F_j \notin S(\mathbf{d}^m) \text{ and } F_j + s \in S(\mathbf{d}^m), \text{ for } \forall s \in S(\mathbf{d}^m) \setminus \{0\}, j \leq t(\mathbf{d}^m)\}, \quad (2)$$

and  $F_j \neq F_k$  if  $j \neq k$ . Set  $S'(\mathbf{d}^m)$  is not empty since  $F(\mathbf{d}^m) \in S'(\mathbf{d}^m)$  for any minimal generating set  $(d_1, \dots, d_m)$ .

The semigroup ring  $k[X_1, \dots, X_m]$  over a field  $k$  of characteristic 0 associated with  $S(\mathbf{d}^m)$  is a polynomial subring graded by  $\deg X_i = d_i$ ,  $i = 1, \dots, m$ , and generated by all monomials  $z^{d_i}$ . The Hilbert series  $H(\mathbf{d}^m; z)$  of a graded subring  $k[z^{d_1}, \dots, z^{d_m}]$  is a rational function [11]

$$H(\mathbf{d}^m; z) = \frac{Q(\mathbf{d}^m; z)}{\prod_{j=1}^m (1 - z^{d_j})}, \quad (3)$$

where  $H(\mathbf{d}^m; z)$  has a pole  $z = 1$  of order 1. The numerator  $Q(\mathbf{d}^m; z)$  is a polynomial in  $z$ ,

$$Q(\mathbf{d}^m; z) = 1 - Q_1(\mathbf{d}^m; z) + Q_2(\mathbf{d}^m; z) - \dots + (-1)^{m-1} Q_{m-1}(\mathbf{d}^m; z), \quad \Sigma_m = \sum_{k=1}^m d_k, \quad (4)$$

$$Q_{m-1}(\mathbf{d}^m; z) = Q'_{m-1}(\mathbf{d}^m; z) + z^{F(\mathbf{d}^m) + \Sigma_m}, \quad \deg Q'_{m-1}(\mathbf{d}^m; z) < F(\mathbf{d}^m) + \Sigma_m. \quad (5)$$

$$Q_i(\mathbf{d}^m; z) = \sum_{j=1}^{\beta_i(\mathbf{d}^m)} z^{C_{j,i}}, \quad 1 \leq i \leq m-1, \quad \deg Q_i(\mathbf{d}^m; z) < \deg Q_{i+1}(\mathbf{d}^m; z). \quad (6)$$

In formula (6) the numbers  $C_{j,i}$  and  $\beta_i(\mathbf{d}^m)$  denote *the syzygy degrees* and *the Betti numbers*, respectively. The summands  $z^{C_{j,i}}$  in (6) stand for the syzygies of different kinds and  $C_{j,i}$  are the degrees of homogeneous basic invariants for the syzygies of the  $i$ th kind,

$$\begin{aligned} C_{j,i} \in \mathbb{N}, \quad C_{j+1,i} \geq C_{j,i}, \quad C_{\beta_{i+1},i+1} > C_{\beta_i,i}, \quad C_{1,i+1} > C_{1,i}, \quad \text{and} \\ C_{j,i} \neq C_{r,i+2k-1}, \quad 1 \leq j \leq \beta_i(\mathbf{d}^m), \quad 1 \leq r \leq \beta_{i+2k-1}(\mathbf{d}^m), \quad 1 \leq k \leq \left\lfloor \frac{m-i}{2} \right\rfloor. \end{aligned} \quad (7)$$

The last requirement (7) means that all necessary cancellations (annihilations) of terms  $z^{C_{j,i}}$  in (4) are already performed. However the other equalities,  $C_{j,i} = C_{r,i+2k}$  and  $C_{j,i} = C_{q,i}$ ,  $j \neq q$ , are not forbidden excluding the syzygy degrees of the last  $(m-1)$ th kind [4]. The numbers of terms  $z^{C_{j,i}}$  in summands are determined by  $\beta_i(\mathbf{d}^m)$  which satisfy the equality [11]

$$1 - \beta_1(\mathbf{d}^m) + \beta_2(\mathbf{d}^m) - \dots + (-1)^{m-1} \beta_{m-1}(\mathbf{d}^m) = 0. \quad (8)$$

The Betti numbers  $\beta_i(\mathbf{d}^m)$  satisfy also an inequality (see [4], Theorem 7),

$$1 + \beta_1(\mathbf{d}^m) + \beta_2(\mathbf{d}^m) + \dots + \beta_{m-1}(\mathbf{d}^m) \leq d_1 2^{m-1} - 2(m-1). \quad (9)$$

Following [4] denote by  $\mathbb{B}_i(\mathbf{d}^m)$  the set of degrees of the terms  $z^{C_{j,i}}$  (up to degeneration,  $C_{j,i} = C_{q,i}$ ,  $j \neq q$ ) for syzygies of the  $i$ th kind which are entering  $Q_i(\mathbf{d}^m; z)$  in (6),

$$\mathbb{B}_i(\mathbf{d}^m) = \{C_{j,i} \in \mathbb{N} \mid z^{C_{j,i}} \in Q_i(\mathbf{d}^m; z), \quad 1 \leq j \leq \beta_i(\mathbf{d}^m)\}. \quad (10)$$

A containment ( $\in$ ) in (10) means that a monomial  $z^{C_{j,i}}$  enters polynomial  $Q_i(\mathbf{d}^m; z)$  at least once.

By (7) we conclude that any two sets  $\mathbb{B}_i(\mathbf{d}^m)$  and  $\mathbb{B}_q(\mathbf{d}^m)$ , whose indices differ by odd number,  $|q-i| = 2k-1$ , are disjointed, i.e.

$$\mathbb{B}_i(\mathbf{d}^m) \cap \mathbb{B}_{i+2k-1}(\mathbf{d}^m) = \emptyset, \quad 1 \leq i \leq m-1, \quad 1 \leq k \leq \left\lfloor \frac{m-i}{2} \right\rfloor. \quad (11)$$

Note that the relation  $\mathbb{B}_i(\mathbf{d}^m) \cap \mathbb{B}_{i+2k}(\mathbf{d}^m) \neq \emptyset$  is not forbidden. Let  $\oplus$  denote *a sumset* of the finite set  $\mathbb{U} \subset \mathbb{N}$  of integers  $u_p$  with an integer  $\alpha$ ,  $\mathbb{U} \oplus \{\alpha\} = \{u_p + \alpha \mid u_p \in \mathbb{U}\}$ . Then we have,

**Lemma 1** ([4], Lemma 1) *The following equality holds*

$$\mathbb{B}_{m-1}(\mathbf{d}^m) = S'(\mathbf{d}^m) \oplus \{\Sigma_m\}. \quad (12)$$

By consequence of (2) and (12) we get the known equality [8],  $\beta_{m-1}(\mathbf{d}^m) = t(\mathbf{d}^m)$ .

## 2 Two Sorts of Gaps in Numerical Semigroups

Following [6] decompose the set of gaps  $\Delta(\mathbf{d}^m)$  into two sets  $\Delta_{\mathcal{G}}(\mathbf{d}^m)$  and  $\Delta_{\mathcal{H}}(\mathbf{d}^m)$ ,

$$\begin{aligned}\Delta_{\mathcal{G}}(\mathbf{d}^m) &= \{g \notin S(\mathbf{d}^m) \mid F(\mathbf{d}^m) - g \in S(\mathbf{d}^m)\}, \quad \#\Delta_{\mathcal{G}}(\mathbf{d}^m) = c(\mathbf{d}^m) - G(\mathbf{d}^m), \\ \Delta_{\mathcal{H}}(\mathbf{d}^m) &= \{h \notin S(\mathbf{d}^m) \mid F(\mathbf{d}^m) - h \notin S(\mathbf{d}^m)\}, \quad \#\Delta_{\mathcal{H}}(\mathbf{d}^m) = 2G(\mathbf{d}^m) - c(\mathbf{d}^m).\end{aligned}\quad (13)$$

The following Theorem is essential in this article.

**Theorem 1** ([4], Theorem 1) *Let the numerical semigroup  $S(\mathbf{d}^m)$  be given with its Hilbert series  $H(\mathbf{d}^m; z)$ . Then the generating functions for the sets  $\Delta_{\mathcal{H}}(\mathbf{d}^m)$  and  $\Delta_{\mathcal{G}}(\mathbf{d}^m)$  are given by*

$$\begin{aligned}\sum_{h \in \Delta_{\mathcal{H}}(\mathbf{d}^m)} z^h &= -H(\mathbf{d}^m; z) - H(\mathbf{d}^m; z^{-1}) \cdot z^{F(\mathbf{d}^m)}, \\ \sum_{g \in \Delta_{\mathcal{G}}(\mathbf{d}^m)} z^g &= \frac{1}{1-z} + H(\mathbf{d}^m; z^{-1}) \cdot z^{F(\mathbf{d}^m)}.\end{aligned}\quad (14)$$

The next Lemma establishes relationship between the sets,  $S'(\mathbf{d}^m)$  and  $\Delta_{\mathcal{H}}(\mathbf{d}^m)$  for general numerical semigroups, and also gives a basis for definition of symmetric and almost symmetric semigroups.

**Lemma 2** ([1] and [4], Lemma 5) *Let a numerical semigroup  $S(\mathbf{d}^m)$  be given. Then*

$$S'(\mathbf{d}^m) \setminus \{F(\mathbf{d}^m)\} \subseteq \Delta_{\mathcal{H}}(\mathbf{d}^m). \quad (15)$$

### 2.1 Symmetric, pseudosymmetric and almost symmetric semigroups

Imposing requirements on the set  $\Delta_{\mathcal{H}}(\mathbf{d}^m)$  one can simplify significantly the structure of semigroup. A semigroup  $S(\mathbf{d}^m)$  is called *symmetric* if  $\Delta_{\mathcal{H}}(\mathbf{d}^m) = \emptyset$  that by (15) implies  $t(\mathbf{d}^m) = 1$ . By Theorem 1 the following duality relation holds for symmetric semigroups (see [4], Corollary 1),

$$H(\mathbf{d}^m; z) + H(\mathbf{d}^m; z^{-1}) \cdot z^{F(\mathbf{d}^m)} = 0. \quad (16)$$

Notably, all semigroups  $S(\mathbf{d}^2)$  are symmetric. For  $m \geq 3$  the necessary conditions for the minimal set  $\mathbf{d}^m$  to generate a symmetric semigroup were given in [13], Lemma 1.

Another simplification comes if  $F(\mathbf{d}^m)$  is an even number, and  $\Delta_{\mathcal{H}}(\mathbf{d}^m) = \{F(\mathbf{d}^m)/2\}$ . These semigroups are called *pseudosymmetric* and the corresponding duality relation reads

$$H(\mathbf{d}^m; z) + H(\mathbf{d}^m; z^{-1}) \cdot z^{F(\mathbf{d}^m)} + z^{\frac{1}{2}F(\mathbf{d}^m)} = 0. \quad (17)$$

Pseudosymmetric semigroups have necessarily  $t(\mathbf{d}^m) = 2$ , but the opposite statement (sufficient condition) is not true. For  $m = 3$ , the structure of the minimal triple  $\mathbf{d}^3$  generating a pseudosymmetric semigroup was given independently in [12] and [4], Theorem 9. Note that both relations, (16) and (17), are self-dual under transformation  $z \rightarrow z^{-1}$ .

The *almost symmetric semigroups* were introduced in [1] as a generalization of the symmetric and pseudosymmetric ones. They are defined by a set equality in (15),

$$S'(\mathbf{d}^m) \setminus \{F(\mathbf{d}^m)\} = \Delta_{\mathcal{H}}(\mathbf{d}^m) . \quad (18)$$

Equivalence of (18) and another equality,  $t(\mathbf{d}^m) = 1 + \#\Delta_{\mathcal{H}}(\mathbf{d}^m)$ , was proven in [1]. For  $m = 3$ , almost symmetric and pseudosymmetric semigroups coincide. A minimal set  $\mathbf{d}^m$  of special kind generating an almost symmetric semigroup is given in the next Proposition.

**Proposition 1** ([1], Proposition 11) *Let a numerical semigroup  $S(\mathbf{d}^{t+1})$  of maximal edim is generated by tuple  $(t+1, t+1+\frac{g}{t}, t+1+2\frac{g}{t}, \dots, t+1+g)$ , where  $t \geq 1$ ,  $g \geq -1$  such that  $t \mid g$  and  $\gcd(t+1, \frac{g}{t}) = 1$ . Then  $S(\mathbf{d}^{t+1})$  is almost symmetric semigroup with  $S'(\mathbf{d}^{t+1}) = \{\frac{g}{t}, 2\frac{g}{t}, \dots, g\}$ .*

In section 7 we consider the almost symmetric semigroups with maximal edim of generic kind (not satisfying Proposition 1) and give the necessary and efficient conditions for minimal set  $\mathbf{d}^m$  to generate such semigroups.

### 3 Duality Relation for Almost Symmetric Semigroups

Continuing a similar description of symmetric, pseudosymmetric and almost symmetric semigroups we derive here the duality relation for the Hilbert series  $H(\mathbf{d}^m; z)$  for the last ones. By Lemma 1 and definition (18) we have

$$\mathbb{B}_{m-1}(\mathbf{d}^m) = [\Delta_{\mathcal{H}}(\mathbf{d}^m) \cup \{F(\mathbf{d}^m)\}] \oplus \{\Sigma_m\} . \quad (19)$$

Following [3], introduce two functions,  $\tau$  and its inverse  $\tau^{-1}$ , where  $\tau$  maps each polynomial  $\Psi(z) = \sum c_k z^k \in \mathbb{N}[z]$  with  $c_k \in \{0, 1\}$  onto the set of degrees  $\mathbb{K} = \{k \in \mathbb{N} \mid c_k \neq 0\}$ . Since all coefficients of the polynomial  $\Psi(z)$  are 1 or 0, we can uniquely reconstruct a set  $\mathbb{K}$  and vice versa. In this sense  $\tau$  is an isomorphic map. The map  $\tau$  is also linear in the following sense (see [3], Ch. 5):

$$\text{If } \mathbb{U}_1, \mathbb{U}_2 \subset \mathbb{N}, \mathbb{U}_1 \cap \mathbb{U}_2 = \emptyset, \text{ then } \tau^{-1}[\mathbb{U}_1 \cup \mathbb{U}_2] = \tau^{-1}[\mathbb{U}_1] + \tau^{-1}[\mathbb{U}_2] . \quad (20)$$

In particular, by consequence of (10) and Lemma 1 we have

$$\tau^{-1}[\mathbb{B}_{m-1}(\mathbf{d}^m)] = Q_{m-1}(\mathbf{d}^m; z) . \quad (21)$$

**Theorem 2** *Let a semigroup  $S(\mathbf{d}^m)$  be almost symmetric. Then its Hilbert series  $H(\mathbf{d}^m; z)$  satisfies the duality relation*

$$H(\mathbf{d}^m; z) + H(\mathbf{d}^m; z^{-1}) \cdot z^{F(\mathbf{d}^m)} + Q'_{m-1}(\mathbf{d}^m; z) \cdot z^{-\Sigma_m} = 0. \quad (22)$$

where  $Q'_{m-1}(\mathbf{d}^m; z)$  is defined in (5).

**Proof** Acting on the left hand side (*l.h.s.*) and right hand side (*r.h.s.*) of (19) by  $\tau^{-1}$  and applying (20) and (21) we get

$$z^{-\Sigma_m} \cdot Q_{m-1}(\mathbf{d}^m; z) = z^{F(\mathbf{d}^m)} + \sum_{h \in \Delta_{\mathcal{H}}(\mathbf{d}^m)} z^h. \quad (23)$$

Combining (23) with (14) we obtain,

$$H(\mathbf{d}^m; z) + H(\mathbf{d}^m; z^{-1}) \cdot z^{F(\mathbf{d}^m)} = z^{F(\mathbf{d}^m)} - Q_{m-1}(\mathbf{d}^m; z) \cdot z^{-\Sigma_m}.$$

Substituting the representation (5) for  $Q_{m-1}(\mathbf{d}^m; z)$  into the last equation we come to the duality relation (22) for the Hilbert series for the almost symmetric semigroups.  $\square$

Require that relation (22) be self-dual under transformation  $z \rightarrow z^{-1}$ , apply it to (22) and get

$$H(\mathbf{d}^m; z^{-1}) + H(\mathbf{d}^m; z) \cdot z^{-F(\mathbf{d}^m)} + Q'_{m-1}(\mathbf{d}^m; z^{-1}) \cdot z^{\Sigma_m} = 0. \quad (24)$$

By comparison of (22) and (24) we obtain a necessary condition for  $S(\mathbf{d}^m)$  to be almost symmetric,

$$Q'_{m-1}(\mathbf{d}^m; z) \cdot z^{-\Sigma_m} = Q'_{m-1}(\mathbf{d}^m; z^{-1}) \cdot z^{F(\mathbf{d}^m) + \Sigma_m}. \quad (25)$$

The last equation has clear explanation. Indeed, consider  $Q'_{m-1}(\mathbf{d}^m; z)$  which has according to (5) and Lemma 1 the following form,  $Q'_{m-1}(\mathbf{d}^m; z) = \sum_{j=1}^{t(\mathbf{d}^m)-1} z^{F_j + \Sigma_m}$ , and substitute it into (25),

$$\sum_{j=1}^{t(\mathbf{d}^m)-1} z^{F_j} = \sum_{j=1}^{t(\mathbf{d}^m)-1} z^{-F_j + F(\mathbf{d}^m)}. \quad (26)$$

Formula (26) is equivalent to the following sequence of equalities,

$$F_{u_1} = F(\mathbf{d}^m) - F_{v_1}, \quad F_{u_2} = F(\mathbf{d}^m) - F_{v_2}, \dots, \quad F_{u_{t-1}} = F(\mathbf{d}^m) - F_{v_{t-1}},$$

where  $u_1, \dots, u_{t-1}$  and  $v_1, \dots, v_{t-1}$  account for different arrangements of the set  $S'(\mathbf{d}^m) \setminus \{F(\mathbf{d}^m)\}$ ,

$$\{F_{u_1}, \dots, F_{u_{t-1}}\} \equiv \{F_{v_1}, \dots, F_{v_{t-1}}\} \equiv S'(\mathbf{d}^m) \setminus \{F(\mathbf{d}^m)\}, \quad 1 \leq u_i, v_i \leq t(\mathbf{d}^m) - 1.$$

**Corollary 1** A bijection  $\{F_{u_1}, \dots, F_{u_{t-1}}\} \iff \{F_{u_1}, \dots, F_{u_{t-1}}\}$  does have one fixed pont  $\frac{1}{2}F(\mathbf{d}^m)$  iff  $t(\mathbf{d}^m)$  is an even number, and has not fixed pont iff  $t(\mathbf{d}^m)$  is an odd number.

One can make one step further and find the duality relation for the numerator  $Q(\mathbf{d}^m; z)$  of the Hilbert series of the almost symmetric semigroups. This seams to be reasonable because of the lower syzygies terms  $z^{C_{j,i}}$ ,  $i \leq m-2$ , which are left behind the relation (25).

**Theorem 3** Let the numerical semigroup  $S(\mathbf{d}^m)$  be almost symmetric. Then the numerator  $Q(\mathbf{d}^m; z)$  of its Hilbert series satisfies the duality relation,

$$Q(\mathbf{d}^m; z) + (-1)^m Q(\mathbf{d}^m; z^{-1}) \cdot z^{F(\mathbf{d}^m) + \Sigma_m} + Q'_{m-1}(\mathbf{d}^m; z) \cdot z^{-\Sigma_m} \prod_{j=1}^m (1 - z^{d_j}) = 0. \quad (27)$$

**Proof** Make use of representation (3) for the Hilbert series  $H(\mathbf{d}^m; z)$  and insert it into (22). Multiplying the r.h.s. and l.h.s. of obtained equation by  $\prod_{j=1}^m (1 - z^{d_j})$  and keeping in mind the identity  $\prod_{j=1}^m (1 - z^{d_j}) / (1 - z^{-d_j}) = (-1)^m z^{\Sigma_m}$  we arrive at

$$Q(\mathbf{d}^m; z) + (-1)^m Q(\mathbf{d}^m; z^{-1}) \cdot z^{F(\mathbf{d}^m) + \Sigma_m} = \prod_{j=1}^m (1 - z^{d_j}) \left[ z^{F(\mathbf{d}^m)} - Q_{m-1}(\mathbf{d}^m; z) \cdot z^{-\Sigma_m} \right]. \quad (28)$$

Substituting (5) for  $Q_{m-1}(\mathbf{d}^m; z)$  into the r.h.s. of (28) and simplifying it we get (27).  $\square$

For short, following [6] denote  $\#\Delta_{\mathcal{H}}(\mathbf{d}^m) = \gamma(\mathbf{d}^m)$  and by consequence of (4) write another form of (27) which is more suitable to deal with,

$$\begin{aligned} & \sum_{r=1}^{\gamma(\mathbf{d}^m)} z^{F_r} \cdot \left( \underbrace{1}_{\beta_1(\mathbf{d}^m)} - \sum_{j=1}^m z^{d_j} + \sum_{j>k=1}^m z^{d_j+d_k} - \dots - (-1)^m \sum_{j=1}^m z^{\Sigma_m-d_j} + \underbrace{(-1)^m z^{\Sigma_m}}_{\beta_2(\mathbf{d}^m)} \right) + \\ & \underbrace{1}_{\beta_1(\mathbf{d}^m)} - \sum_{j=1}^{\beta_1(\mathbf{d}^m)} z^{C_{j,1}} + \sum_{j=1}^{\beta_2(\mathbf{d}^m)} z^{C_{j,2}} - \dots + (-1)^{m-1} \left( \sum_{r=1}^{\gamma(\mathbf{d}^m)} \underbrace{z^{F_r+\Sigma_m}}_{\beta_3(\mathbf{d}^m)} + \underbrace{z^{F(\mathbf{d}^m)+\Sigma_m}}_{\beta_{m-2}(\mathbf{d}^m)} \right) + \\ & (-1)^m \left[ \underbrace{z^{F(\mathbf{d}^m)+\Sigma_m}}_{\beta_1(\mathbf{d}^m)} - \sum_{j=1}^{\beta_1(\mathbf{d}^m)} z^{-C_{j,1}+F(\mathbf{d}^m)+\Sigma_m} + \dots + (-1)^{m-1} \left( \sum_{r=1}^{\gamma(\mathbf{d}^m)} \underbrace{z^{F(\mathbf{d}^m)-F_r}}_{\beta_3(\mathbf{d}^m)} + \underbrace{1}_{\beta_{m-2}(\mathbf{d}^m)} \right) \right] = 0. \end{aligned}$$

In the last equation we have underbraced  $4\gamma(\mathbf{d}^m) + 4$  terms which are cancelling pairwise. After this simplification and further recasting of the rest of terms we get finally,

$$\begin{aligned} & \sum_{r=1}^{\gamma(\mathbf{d}^m)} \left[ \sum_{j=1}^m z^{d_j+F_r} - \sum_{j>k=1}^m z^{d_j+d_k+F_r} + \dots + (-1)^m \sum_{j=1}^m z^{\Sigma_m-d_j+F_r} \right] + \\ & \sum_{j=1}^{\beta_1(\mathbf{d}^m)} z^{C_{j,1}} - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} z^{C_{j,2}} + \sum_{j=1}^{\beta_3(\mathbf{d}^m)} z^{C_{j,3}} - \dots + (-1)^{m-1} \sum_{j=1}^{\beta_{m-2}(\mathbf{d}^m)} z^{C_{j,m-2}} = \\ & (-1)^{m-1} \left[ \sum_{j=1}^{\beta_1(\mathbf{d}^m)} z^{\bar{C}_{j,1}} - \sum_{j=1}^{\beta_2(\mathbf{d}^m)} z^{\bar{C}_{j,2}} + \dots + (-1)^{m-1} \sum_{j=1}^{\beta_{m-2}(\mathbf{d}^m)} z^{\bar{C}_{j,m-2}} \right]. \end{aligned} \quad (29)$$

The degrees  $\overline{C}_{j,i} = -C_{j,i} + F(\mathbf{d}^m) + \Sigma_m$  compose another numerical set  $\overline{\mathbb{B}}_i(\mathbf{d}^m)$  similar to  $\mathbb{B}_i(\mathbf{d}^m)$ ,

$$\overline{\mathbb{B}}_i(\mathbf{d}^m) = \{ \overline{C}_{j,i} \in \mathbb{N} \mid z^{C_{j,i}} \in Q_i(\mathbf{d}^m; z), 1 \leq j \leq \beta_i(\mathbf{d}^m) \} . \quad (30)$$

Equation (29) is a master equation for study the structure of syzygies for the almost symmetric semigroups. In the rest of this paper the main problem is considered: what can be said about indeterminate degrees  $C_{j,i}$  in Eq. (29), or more precisely, how to find them in terms of the given generators  $d_i$  and gaps  $F_j \in \Delta_{\mathcal{H}}(\mathbf{d}^m)$ .

A naive approach shows that, in order to satisfy Eq. (29), we have to recast its terms in such a way that in every its l.h.s. and r.h.s. would remain only positive terms, and after that to equate all degrees of the terms in new recasting equation in its l.h.s. and r.h.s.. In other words, we have to build a bijection between two multisets, a multiset  $\mathfrak{M}$  of given gaps  $F_j$ , generators  $d_i$  and their linear combinations, and a multiset  $\mathfrak{X}$  of indeterminate degrees  $C_{j,i}$  and  $\overline{C}_{j,i}$ .

## 4 Multisets and Multiset Operations

Making preliminary preparation we start with concept of a multiset and basic multiset operations allowed by this structure (see [9], [5] and references therein). More rigorous analysis of Eq. (29) and associated multisets follows later in this and next sections 5 and 6.

A concept of multiset is a generalization of the concept of a set. A member of a multiset can have more than one occurrence (called multiplicity, don't confuse with multiplicity of semigroup), while each member of a set occurs once. A privileged role is still given to (ordinary) sets when defining maps, as there is no clear notion of maps (functions) between multisets.

Consider a multiset which can be formally defined as a pair  $\langle \mathbb{K}, \sigma_{\mathbb{K}} \rangle$  where  $\mathbb{K}$  is some finite set,  $\#\mathbb{K} < \infty$ , and  $\sigma_{\mathbb{K}} : \mathbb{K} \mapsto \mathbb{N}$  is a function from  $\mathbb{K}$  to the set of positive integers. We call  $\langle \mathbb{K}, \sigma_{\mathbb{K}} \rangle$  a *standard representation* of multiset. For each  $\omega \in \mathbb{K}$  the multiplicity (that is, number of occurrences) of  $\omega$  is the number  $\sigma_{\mathbb{K}}(\omega) \geq 0$  such that

$$\#\langle \mathbb{K}, \sigma_{\mathbb{K}} \rangle = \sum_{\omega \in \mathbb{K}} \sigma_{\mathbb{K}}(\omega) , \quad \text{and by definition if } \omega \notin \mathbb{K} \text{ then } \sigma_{\mathbb{K}}(\omega) = 0 . \quad (31)$$

We say that element  $(\omega, \sigma_{\mathbb{K}}(\omega))$  belongs to multiset  $\langle \mathbb{K}, \sigma_{\mathbb{K}} \rangle$  iff  $\sigma_{\mathbb{K}}(\omega)$  takes a positive value, i.e.

$$(\omega, \sigma_{\mathbb{K}}(\omega)) \in \langle \mathbb{K}, \sigma_{\mathbb{K}} \rangle \text{ iff } \sigma_{\mathbb{K}}(\omega) > 0 , \quad \text{and} \quad (\omega, \sigma_{\mathbb{K}}(\omega)) \notin \langle \mathbb{K}, \sigma_{\mathbb{K}} \rangle \text{ iff } \sigma_{\mathbb{K}}(\omega) = 0 . \quad (32)$$

Let two multisets  $\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle$  and  $\langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle$  be given with functions  $\sigma_{\mathbb{K}_i} : \mathbb{K}_i \mapsto \mathbb{N}$ ,  $i = 1, 2$ . We say also that the following multiset containment ( $\sqsubseteq$ ) holds

$$\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \sqsubseteq \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle , \quad \text{if } \mathbb{K}_1 \subseteq \mathbb{K}_2 \text{ and } \sigma_{\mathbb{K}_1}(\omega) \leq \sigma_{\mathbb{K}_2}(\omega) , \quad \text{for } \forall \omega \in \mathbb{K}_1, \mathbb{K}_2 . \quad (33)$$



In particular, define the multiset equality,

$$\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle = \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle, \quad \text{if } \mathbb{K}_1 = \mathbb{K}_2 \quad \text{and} \quad \sigma_{\mathbb{K}_1}(\omega) = \sigma_{\mathbb{K}_2}(\omega), \quad \text{for } \forall \omega \in \mathbb{K}_1, \mathbb{K}_2. \quad (34)$$

We say that a multiset  $\langle \mathbb{K}, \sigma_{\mathbb{K}} \rangle$  is empty and denote it by  $\langle \mathbb{K}, \widehat{0} \rangle$  if the following equality holds,

$$\sigma_{\mathbb{K}}(\omega) = 0 \quad \text{for } \forall \omega \in \mathbb{K}, \quad \text{i.e.} \quad \sigma_{\mathbb{K}} = \widehat{0} : \mathbb{K} \mapsto \{0\}, \quad (35)$$

and put for empty set  $\mathbb{K} = \emptyset$  by definition  $\sigma_{\emptyset} : \emptyset \mapsto \{0\}$ , i.e.  $\sigma_{\emptyset} = \widehat{0}$  and a multiset  $\langle \emptyset, \sigma_{\emptyset} \rangle$  is empty.

Following [5], denote by  $\bigvee$  the *join* operation of two multisets  $\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle$  and  $\langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle$ ,

$$\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \bigvee \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle = \langle \mathbb{K}_1 \cup \mathbb{K}_2, \sigma_{\mathbb{K}_1 \cup \mathbb{K}_2} \rangle, \quad \sigma_{\mathbb{K}_1 \cup \mathbb{K}_2}(\omega) = \sigma_{\mathbb{K}_1}(\omega) + \sigma_{\mathbb{K}_2}(\omega), \quad (36)$$

so that by (33) and (36) we get  $\langle \mathbb{K}_i, \sigma_{\mathbb{K}_i} \rangle \subseteq \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \bigvee \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle$ ,  $i = 1, 2$  and

$$\# \left[ \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \bigvee \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \right] = \# \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle + \# \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle.$$

If a multiset  $\langle \mathbb{K}_2, \widehat{0} \rangle$  is empty, then by (35) and (36) we have  $\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle = \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \bigvee \langle \mathbb{K}_2, \widehat{0} \rangle = \langle \mathbb{K}_2, \widehat{0} \rangle \bigvee \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle$ , that encompasses also the case  $\mathbb{K}_2 = \emptyset$ . By the  $\bigvee$  operation a multiset  $\langle \mathbb{K}, \sigma_{\mathbb{K}} \rangle$  and an element  $(\omega, \sigma_{\mathbb{K}_1 \cup \mathbb{K}_2}(\omega))$  can be represented as follows,

$$\langle \mathbb{K}, \sigma_{\mathbb{K}} \rangle = \bigvee_{\omega \in \mathbb{K}} (\omega, \sigma_{\mathbb{K}}(\omega)), \quad (\omega, \sigma_{\mathbb{K}_1}(\omega) + \sigma_{\mathbb{K}_2}(\omega)) = (\omega, \sigma_{\mathbb{K}_1}(\omega)) \bigvee (\omega, \sigma_{\mathbb{K}_2}(\omega)). \quad (37)$$

By consequence of (36) the  $\bigvee$  - operation satisfies the commutative and associative laws,

$$\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \bigvee \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle = \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \bigvee \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle, \quad (38)$$

$$\left[ \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \bigvee \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \right] \bigvee \langle \mathbb{K}_3, \sigma_{\mathbb{K}_3} \rangle = \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \bigvee \left[ \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \bigvee \langle \mathbb{K}_3, \sigma_{\mathbb{K}_3} \rangle \right] = \bigvee_{j=1}^3 \langle \mathbb{K}_j, \sigma_{\mathbb{K}_j} \rangle.$$

Let two multisets  $\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle$  and  $\langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle$  be given such that  $\mathbb{K}_1, \mathbb{K}_2 \subseteq \mathbb{N}$ . i.e. if  $\omega \in \mathbb{K}_1$  and  $\xi \in \mathbb{K}_2$  then  $\omega + \xi \in \mathbb{N}$ . Denote by  $\widehat{\oplus}$  their *join sumset* operation and define it as follows,

$$\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \widehat{\oplus} \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle =: \bigvee_{\substack{\omega \in \mathbb{K}_1 \\ \xi \in \mathbb{K}_2}} [(\omega, \sigma_{\mathbb{K}_1}(\omega)) \oplus (\xi, \sigma_{\mathbb{K}_2}(\xi))] = \bigvee_{\substack{\omega \in \mathbb{K}_1 \\ \xi \in \mathbb{K}_2}} (\omega + \xi, \sigma_{\mathbb{K}_1}(\omega) \cdot \sigma_{\mathbb{K}_2}(\xi)), \quad (39)$$

$$\# \left[ \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \widehat{\oplus} \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \right] = \# \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \cdot \# \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle, \quad (40)$$

where  $\oplus$  is a usual sumset operation which was already used in (12) and (19). By (39) the  $\widehat{\oplus}$  operation satisfies the commutative law,

$$\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \widehat{\oplus} \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle = \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \widehat{\oplus} \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle. \quad (41)$$

Distributive law of the  $\widehat{\oplus}$  - operation holds over the  $\vee$  - operation,

$$\left[ \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \vee \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \right] \widehat{\oplus} \langle \mathbb{K}_3, \sigma_{\mathbb{K}_3} \rangle = \left[ \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \widehat{\oplus} \langle \mathbb{K}_3, \sigma_{\mathbb{K}_3} \rangle \right] \vee \left[ \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \widehat{\oplus} \langle \mathbb{K}_3, \sigma_{\mathbb{K}_3} \rangle \right]. \quad (42)$$

We prove (42) making use of (36), (37) and (39) and start with its l.h.s. in the form,

$$\begin{aligned} \bigvee_{\omega \in \mathbb{K}_1 \cup \mathbb{K}_2, \xi \in \mathbb{K}_3} (\omega, \sigma_{\mathbb{K}_1 \cup \mathbb{K}_2}(\omega)) \oplus (\xi, \sigma_{\mathbb{K}_3}(\xi)) &= \bigvee_{\omega \in \mathbb{K}_1 \cup \mathbb{K}_2, \xi \in \mathbb{K}_3} (\omega + \xi, [\sigma_{\mathbb{K}_1}(\omega) + \sigma_{\mathbb{K}_2}(\omega)] \cdot \sigma_{\mathbb{K}_3}(\xi)) \\ &= \left[ \bigvee_{\omega \in \mathbb{K}_1, \xi \in \mathbb{K}_3} (\omega + \xi, \sigma_{\mathbb{K}_1}(\omega) \cdot \sigma_{\mathbb{K}_3}(\xi)) \right] \vee \left[ \bigvee_{\omega \in \mathbb{K}_2, \xi \in \mathbb{K}_3} (\omega + \xi, \sigma_{\mathbb{K}_2}(\omega) \cdot \sigma_{\mathbb{K}_3}(\xi)) \right] \\ &= \left[ \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \widehat{\oplus} \langle \mathbb{K}_3, \sigma_{\mathbb{K}_3} \rangle \right] \vee \left[ \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \widehat{\oplus} \langle \mathbb{K}_3, \sigma_{\mathbb{K}_3} \rangle \right]. \end{aligned}$$

Define an intersection of two multisets and denote it by  $\bigwedge$ ,

$$\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \bigwedge \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle = \langle \mathbb{K}_1 \cap \mathbb{K}_2, \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2} \rangle, \quad \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2}(\omega) = \min \{ \sigma_{\mathbb{K}_1}(\omega), \sigma_{\mathbb{K}_2}(\omega) \}, \quad (43)$$

so that by (33) and (43) a following containment holds,

$$\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \bigwedge \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \subseteq \langle \mathbb{K}_i, \sigma_{\mathbb{K}_i} \rangle, \quad i = 1, 2. \quad (44)$$

Let two multisets  $\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle$  and  $\langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle$  be given such that  $\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \subseteq \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle$  in accordance with (33). Define their set difference  $\setminus$  as follows,

$$\langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \setminus \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle = \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2 \setminus \mathbb{K}_1} \rangle, \quad \sigma_{\mathbb{K}_2 \setminus \mathbb{K}_1}(\omega) = \sigma_{\mathbb{K}_2}(\omega) - \sigma_{\mathbb{K}_1}(\omega), \quad \text{for } \forall \omega \in \mathbb{K}_2. \quad (45)$$

By (36) and (45) it follows,

$$\left[ \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \vee \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \right] \setminus \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle = \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle, \quad \langle \mathbb{K}, \sigma_{\mathbb{K}} \rangle \setminus \langle \mathbb{K}, \sigma_{\mathbb{K}} \rangle = \langle \mathbb{K}, \widehat{0} \rangle. \quad (46)$$

We prove three Lemmas before going to the main Theorem 4 on multiset equalities.

**Lemma 3** *Let three multisets be given such that  $\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \subseteq \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \vee \langle \mathbb{K}_3, \sigma_{\mathbb{K}_3} \rangle$ . Then*

$$\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \subseteq \left[ \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \bigwedge \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \right] \vee \left[ \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \bigwedge \langle \mathbb{K}_3, \sigma_{\mathbb{K}_3} \rangle \right]. \quad (47)$$

**Proof** Start with given containment and make worth of (34) and (36)

$$\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \subseteq \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \vee \langle \mathbb{K}_3, \sigma_{\mathbb{K}_3} \rangle = \langle \mathbb{K}_2 \cup \mathbb{K}_3, \sigma_{\mathbb{K}_2 \cup \mathbb{K}_3} \rangle \rightarrow \begin{cases} \mathbb{K}_1 \subseteq \mathbb{K}_2 \cup \mathbb{K}_3, \\ \sigma_{\mathbb{K}_1}(\omega) \leq \sigma_{\mathbb{K}_2}(\omega) + \sigma_{\mathbb{K}_3}(\omega) \end{cases}. \quad (48)$$

However, the last containment,  $\mathbb{K}_1 \subseteq \mathbb{K}_2 \cup \mathbb{K}_3$  implies a set identity  $\mathbb{K}_1 = [\mathbb{K}_1 \cap \mathbb{K}_2] \cup [\mathbb{K}_1 \cap \mathbb{K}_3]$ .

We write inequality in (48) in more details in three different regions of the set  $\mathbb{K}_1$ ,

$$\text{for } \forall \omega \in \mathbb{K}_1 : \quad \sigma_{\mathbb{K}_1}(\omega) \leq \begin{cases} \sigma_{\mathbb{K}_2}(\omega), & \text{if } \omega \notin \mathbb{K}_1 \cap \mathbb{K}_3, \\ \sigma_{\mathbb{K}_2}(\omega) + \sigma_{\mathbb{K}_3}(\omega), & \text{if } \omega \in \mathbb{K}_1 \cap \mathbb{K}_2 \cap \mathbb{K}_3, \\ \sigma_{\mathbb{K}_3}(\omega), & \text{if } \omega \notin \mathbb{K}_1 \cap \mathbb{K}_2. \end{cases} \quad (49)$$

Apply (36), (43) and identity  $\mathbb{K}_1 = [\mathbb{K}_1 \cap \mathbb{K}_2] \cup [\mathbb{K}_1 \cap \mathbb{K}_3]$  to the r.h.s. of (47),

$$\begin{aligned} \langle \mathbb{K}_1 \cap \mathbb{K}_2, \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2} \rangle \vee \langle \mathbb{K}_1 \cap \mathbb{K}_3, \sigma_{\mathbb{K}_1 \cap \mathbb{K}_3} \rangle &= \langle [\mathbb{K}_1 \cap \mathbb{K}_2] \cup [\mathbb{K}_1 \cap \mathbb{K}_3], \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2} + \sigma_{\mathbb{K}_1 \cap \mathbb{K}_3} \rangle \\ &= \langle \mathbb{K}_1, \mu_{\mathbb{K}_1, \mathbb{K}_2} + \mu_{\mathbb{K}_1, \mathbb{K}_3} \rangle, \quad \text{where } \mu_{\mathbb{K}_i, \mathbb{K}_j} = \min \{ \sigma_{\mathbb{K}_i}(\omega), \sigma_{\mathbb{K}_j}(\omega) \}. \end{aligned} \quad (50)$$

Consider  $\mu_{\mathbb{K}_1, \mathbb{K}_2} + \mu_{\mathbb{K}_1, \mathbb{K}_3}$  in three different regions of the set  $\mathbb{K}_1$ : 1)  $\omega \notin \mathbb{K}_1 \cap \mathbb{K}_3$ , 2)  $\omega \notin \mathbb{K}_1 \cap \mathbb{K}_2$  and 3)  $\omega \in \mathbb{K}_1 \cap \mathbb{K}_2 \cap \mathbb{K}_3$ . By (31) and (49) we have in the first two regions,

$$\begin{aligned} \mu_{\mathbb{K}_1, \mathbb{K}_2} &= \begin{cases} \sigma_{\mathbb{K}_1}(\omega) & \text{if } \omega \notin \mathbb{K}_1 \cap \mathbb{K}_3 \\ 0 & \text{if } \omega \notin \mathbb{K}_1 \cap \mathbb{K}_2 \end{cases}, \quad \mu_{\mathbb{K}_1, \mathbb{K}_3} = \begin{cases} 0 & \text{if } \omega \notin \mathbb{K}_1 \cap \mathbb{K}_3 \\ \sigma_{\mathbb{K}_1}(\omega) & \text{if } \omega \notin \mathbb{K}_1 \cap \mathbb{K}_2 \end{cases}, \\ \mu_{\mathbb{K}_1, \mathbb{K}_2} + \mu_{\mathbb{K}_1, \mathbb{K}_3} &= \sigma_{\mathbb{K}_1}(\omega) \quad \text{if } \omega \notin \mathbb{K}_1 \cap \mathbb{K}_2 \quad \text{or} \quad \omega \notin \mathbb{K}_1 \cap \mathbb{K}_3. \end{aligned} \quad (51)$$

The 3rd region,  $\omega \in \mathbb{K}_1 \cap \mathbb{K}_2 \cap \mathbb{K}_3$ , requires more accurate operation. By inequality (48) we have three options:  $\sigma_{\mathbb{K}_1}(\omega) \leq \sigma_{\mathbb{K}_2}(\omega)$  and  $\sigma_{\mathbb{K}_1}(\omega) \leq \sigma_{\mathbb{K}_3}(\omega)$ , or  $\sigma_{\mathbb{K}_1}(\omega) \leq \sigma_{\mathbb{K}_2}(\omega)$  and  $\sigma_{\mathbb{K}_1}(\omega) \geq \sigma_{\mathbb{K}_3}(\omega)$  or  $\sigma_{\mathbb{K}_1}(\omega) \geq \sigma_{\mathbb{K}_2}(\omega)$  and  $\sigma_{\mathbb{K}_1}(\omega) \leq \sigma_{\mathbb{K}_3}(\omega)$ , so

$$\text{for } \forall \omega \in \mathbb{K}_1 \cap \mathbb{K}_2 \cap \mathbb{K}_3 : \quad \mu_{\mathbb{K}_1, \mathbb{K}_2} + \mu_{\mathbb{K}_1, \mathbb{K}_3} = \begin{cases} 2\sigma_{\mathbb{K}_1}(\omega) & \text{if } \sigma_{\mathbb{K}_1} \leq \sigma_{\mathbb{K}_2}, \sigma_{\mathbb{K}_1} \leq \sigma_{\mathbb{K}_3}, \\ \sigma_{\mathbb{K}_1}(\omega) + \sigma_{\mathbb{K}_3}(\omega) & \text{if } \sigma_{\mathbb{K}_1} \leq \sigma_{\mathbb{K}_2}, \sigma_{\mathbb{K}_1} \geq \sigma_{\mathbb{K}_3}, \\ \sigma_{\mathbb{K}_1}(\omega) + \sigma_{\mathbb{K}_2}(\omega) & \text{if } \sigma_{\mathbb{K}_1} \geq \sigma_{\mathbb{K}_2}, \sigma_{\mathbb{K}_1} \leq \sigma_{\mathbb{K}_3}. \end{cases}$$

Thus, keeping in mind the mid line in (49) we can summarize the last equalities and (51),

$$\text{for } \forall \omega \in \mathbb{K}_1 : \quad \mu_{\mathbb{K}_1, \mathbb{K}_2} + \mu_{\mathbb{K}_1, \mathbb{K}_3} \begin{cases} = \sigma_{\mathbb{K}_1}(\omega), & \text{if } \omega \notin \mathbb{K}_1 \cap \mathbb{K}_3, \\ \geq \sigma_{\mathbb{K}_1}(\omega), & \text{if } \omega \in \mathbb{K}_1 \cap \mathbb{K}_2 \cap \mathbb{K}_3, \\ = \sigma_{\mathbb{K}_1}(\omega), & \text{if } \omega \notin \mathbb{K}_1 \cap \mathbb{K}_2. \end{cases} \quad (52)$$

Substituting (52) into (50) and comparing the obtained multiset with  $\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle$  we arrive at (47) that finishes proof of Lemma.  $\square$

Note that a containment  $\mathbb{K}_1 \subseteq \mathbb{K}_2 \cup \mathbb{K}_3$  implies a set equality  $\mathbb{K}_1 = [\mathbb{K}_1 \cap \mathbb{K}_2] \cup [\mathbb{K}_1 \cap \mathbb{K}_3]$ , but according to Lemma 3 such implication cannot be extended onto multisets.

Before going to the next Lemma show that

$$\text{If } \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \subseteq \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \subseteq \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \quad \text{then} \quad \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle = \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle. \quad (53)$$

Indeed, according to (33) the double containment of multisets implies the double containment of sets,  $\mathbb{K}_1 \subseteq \mathbb{K}_2 \subseteq \mathbb{K}_1$ , and two nonstrict inequalities,  $\sigma_{\mathbb{K}_1}(\omega) \leq \sigma_{\mathbb{K}_2}(\omega) \leq \sigma_{\mathbb{K}_1}(\omega)$ , that gives (53).

**Lemma 4** *Let three multisets be given such that  $\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \subseteq \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \vee \langle \mathbb{K}_3, \sigma_{\mathbb{K}_3} \rangle$ . If a multiset  $\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \wedge \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle$  is empty then*

$$\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle = \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \wedge \langle \mathbb{K}_3, \sigma_{\mathbb{K}_3} \rangle . \quad (54)$$

**Proof** First, since a multiset  $\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \wedge \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle$  is empty, then Lemma 3 and summation law with empty multiset imply a containment  $\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \subseteq \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \wedge \langle \mathbb{K}_3, \sigma_{\mathbb{K}_3} \rangle$ . However, by (44) we have an opposite containment,  $\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \wedge \langle \mathbb{K}_3, \sigma_{\mathbb{K}_3} \rangle \subseteq \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle$ . Combining together both containments and applying (53) we arrive at (54).  $\square$

**Lemma 5** *Let two multisets  $\langle \mathbb{K}_i, \sigma_{\mathbb{K}_i} \rangle$ ,  $i = 1, 2$  be given such that their intersection  $\langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle = \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \wedge \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle$  is not empty. Then*

$$\langle \mathbb{K}_i, \sigma_{\mathbb{K}_i} \rangle = [\langle \mathbb{K}_i, \sigma_{\mathbb{K}_i} \rangle \setminus \langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle] \vee \langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle . \quad (55)$$

and a following multiset  $[\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \setminus \langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle] \wedge [\langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \setminus \langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle]$  is empty,

**Proof** Calculate the r.h.s. of (55) in accordance with definitions of the join ' $\vee$ ' (36), intersection ' $\wedge$ ' (43) and set difference ' $\setminus$ ' (45) operations for multisets. For  $i = 1$  we have

$$\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \setminus \langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle = \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \setminus \langle \mathbb{K}_1 \cap \mathbb{K}_2, \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2} \rangle = \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} - \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2} \rangle . \quad (56)$$

Thus, by consequence of (56) we get

$$[\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \setminus \langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle] \vee \langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle = \langle \mathbb{K}_1 \cup \mathbb{K}_{12}, \sigma_{\mathbb{K}_1} - \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2} + \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2} \rangle = \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle .$$

The proof for  $i = 2$  is similar. Thus, the 1st part of Lemma is proven. As for the 2nd part of Lemma, consider a multiset  $\langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle$  according to (43),

$$\langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle = \langle \mathbb{K}_1 \cap \mathbb{K}_2, \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2} \rangle , \quad \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2}(\omega) = \begin{cases} \sigma_{\mathbb{K}_1}(\omega), & \text{if } \sigma_{\mathbb{K}_1}(\omega) \leq \sigma_{\mathbb{K}_2}(\omega), \\ \sigma_{\mathbb{K}_2}(\omega), & \text{if } \sigma_{\mathbb{K}_1}(\omega) \geq \sigma_{\mathbb{K}_2}(\omega). \end{cases} \quad (57)$$

Then, by (56) and (57) we have

$$\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \setminus \langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle = \langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} - \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2} \rangle , \quad \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \setminus \langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle = \langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} - \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2} \rangle , \quad (58)$$

$$\text{If } \sigma_{\mathbb{K}_1}(\omega) \leq \sigma_{\mathbb{K}_2}(\omega) \text{ then } \sigma_{\mathbb{K}_1} - \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2} = 0 , \quad \sigma_{\mathbb{K}_2} - \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2} = \sigma_{\mathbb{K}_2}(\omega) - \sigma_{\mathbb{K}_1}(\omega) ,$$

$$\text{If } \sigma_{\mathbb{K}_1}(\omega) \geq \sigma_{\mathbb{K}_2}(\omega) \text{ then } \sigma_{\mathbb{K}_1} - \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2} = \sigma_{\mathbb{K}_1}(\omega) - \sigma_{\mathbb{K}_2}(\omega) , \quad \sigma_{\mathbb{K}_2} - \sigma_{\mathbb{K}_1 \cap \mathbb{K}_2} = 0 . \quad (59)$$

Combining (58) and (59) with the  $\wedge$  operation (43) we can calculate an intersection

$$[\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \setminus \langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle] \wedge [\langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \setminus \langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle] = \langle \mathbb{K}_1 \cap \mathbb{K}_2, \hat{0} \rangle ,$$

that means in accordance with (35) that multiset  $[\langle \mathbb{K}_1, \sigma_{\mathbb{K}_1} \rangle \setminus \langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle] \wedge [\langle \mathbb{K}_2, \sigma_{\mathbb{K}_2} \rangle \setminus \langle \mathbb{K}_{12}, \sigma_{\mathbb{K}_{12}} \rangle]$  is empty. Thus, our Lemma is proven completely.  $\square$

In this paper we study two multisets  $\langle \mathbb{M}, \sigma_{\mathbb{M}} \rangle$  and  $\langle \mathbb{X}, \sigma_{\mathbb{X}} \rangle$  described at the end of section 3. As for  $\langle \mathbb{M}, \sigma_{\mathbb{M}} \rangle$ , the multiplicity  $\sigma_{\mathbb{M}}(\omega)$  accounts for the number of elements  $\omega = d_{i_1} + \dots + d_{i_k} + F_r$ ,  $\omega \in \mathbb{M}$ ,  $1 \leq k \leq m-1$ ,  $1 \leq r \leq \gamma(\mathbf{d}^m)$ , of equal values. As for  $\langle \mathbb{X}, \sigma_{\mathbb{X}} \rangle$ , the multiplicity  $\sigma_{\mathbb{X}}(\xi)$  accounts for the number of elements  $\xi = C_{j,i}, \overline{C}_{j,i}$ ,  $\xi \in \mathbb{X}$ ,  $1 \leq i \leq m-1$ ,  $1 \leq j \leq \beta_i(\mathbf{d}^m)$ , of equal values that comes by much more difficult way making use of the Hilbert syzygy theorem [2].

In fact, in this paper we deal mainly with cardinalities of entire multisets  $\langle \mathbb{M}, \sigma_{\mathbb{M}} \rangle$  and  $\langle \mathbb{X}, \sigma_{\mathbb{X}} \rangle$  or of their submultisets. Therefore, for the sake of brevity we will often reduce the designations for multisets  $\langle \mathbb{M}, \sigma_{\mathbb{M}} \rangle = \mathfrak{M}$  and  $\langle \mathbb{X}, \sigma_{\mathbb{X}} \rangle = \mathfrak{X}$ , skipping the underlying sets  $\mathbb{M}$ ,  $\mathbb{X}$  and the mapping functions  $\sigma_{\mathbb{M}}$ ,  $\sigma_{\mathbb{X}}$ . For example, we shall write (42) as follows,  $[\mathfrak{K}_1 \vee \mathfrak{K}_2] \widehat{\oplus} \mathfrak{K}_3 = [\mathfrak{K}_1 \widehat{\oplus} \mathfrak{K}_3] \vee [\mathfrak{K}_2 \widehat{\oplus} \mathfrak{K}_3]$ , where  $\mathfrak{K}_i = \langle \mathbb{K}_i, \sigma_{\mathbb{K}_i} \rangle$ . We hope that such reduction will not mislead the readers.

#### 4.1 Multisets and equation (29)

After recasting the terms of Eq. (29) in such a way that in every its l.h.s. and r.h.s. would remain only positive terms, all degrees in power terms can be arranged in 4 multisets  $\mathfrak{M}_1$ ,  $\mathfrak{M}_2$  and  $\mathfrak{X}_1$ ,  $\mathfrak{X}_2$ ,

$$\sum_{\omega_1 \in \mathbb{M}_1(\mathbf{d}^m)} z^{\omega_1} + \sum_{\xi_1 \in \mathbb{X}_1(\mathbf{d}^m)} z^{\xi_1} = \sum_{\omega_2 \in \mathbb{M}_2(\mathbf{d}^m)} z^{\omega_2} + \sum_{\xi_2 \in \mathbb{X}_2(\mathbf{d}^m)} z^{\xi_2}. \quad (60)$$

Two of these multisets,  $\mathfrak{M}_1 = \langle \mathbb{M}_1, \sigma_{\mathbb{M}_1} \rangle$  and  $\mathfrak{X}_1 = \langle \mathbb{X}_1, \sigma_{\mathbb{X}_1} \rangle$ , are distributed in the l.h.s. of Eq. (29) while the other two,  $\mathfrak{M}_2 = \langle \mathbb{M}_2, \sigma_{\mathbb{M}_2} \rangle$  and  $\mathfrak{X}_2 = \langle \mathbb{X}_2, \sigma_{\mathbb{X}_2} \rangle$ , are distributed in the r.h.s., of Eq. (29). The sets  $\mathbb{M}_1$  and  $\mathbb{M}_2$  are the sets of partial sums  $\omega_{1,2} = d_{i_1} + \dots + d_{i_k} + F_r$  of gaps  $F_j$  and generators  $d_i$ . Both sets  $\mathbb{X}_1$  and  $\mathbb{X}_2$  are the sets of degrees  $\xi_{1,2} = C_{j,i}, \overline{C}_{j,i}$  defined in (7) and (30). In view of definition (36) of the operation  $\vee$  the multiset equality associated with Eq. (60) reads,

$$\mathfrak{M}_1 \vee \mathfrak{X}_1 = \mathfrak{M}_2 \vee \mathfrak{X}_2. \quad (61)$$

Denote by  $\emptyset$  the empty multiset and prove the following theorem on multiset equalities.

**Theorem 4** *Let two finite multisets  $\mathfrak{M}_1$ ,  $\mathfrak{M}_2$  of integers and two finite multisets  $\mathfrak{X}_1$ ,  $\mathfrak{X}_2$  of indeterminate elements be given such that*

$$\mathfrak{M}_1 \wedge \mathfrak{M}_2 = \mathfrak{M}_{12} \neq \emptyset, \quad \mathfrak{X}_1 \wedge \mathfrak{X}_2 = \mathfrak{X}_{12} \neq \emptyset, \quad (62)$$

*and let a multiset equality (61) be given. Then the following hold*

$$\mathfrak{X}_1 \setminus \mathfrak{X}_{12} = \mathfrak{M}_2 \setminus \mathfrak{M}_{12}, \quad \mathfrak{X}_2 \setminus \mathfrak{X}_{12} = \mathfrak{M}_1 \setminus \mathfrak{M}_{12}. \quad (63)$$

**Proof** By consequence of (55) we can represent the multisets  $\mathfrak{M}_1$ ,  $\mathfrak{M}_2$  and  $\mathfrak{X}_1$ ,  $\mathfrak{X}_2$  as follows,

$$\mathfrak{M}_i = [\mathfrak{M}_i \setminus \mathfrak{M}_{12}] \vee \mathfrak{M}_{12} , \quad \mathfrak{X}_i = [\mathfrak{X}_i \setminus \mathfrak{X}_{12}] \vee \mathfrak{X}_{12} , \quad i = 1, 2 , \quad (64)$$

and substitute (64) into (61),

$$[[\mathfrak{M}_1 \setminus \mathfrak{M}_{12}] \vee \mathfrak{M}_{12}] \vee [[\mathfrak{X}_1 \setminus \mathfrak{X}_{12}] \vee \mathfrak{X}_{12}] = [[\mathfrak{M}_2 \setminus \mathfrak{M}_{12}] \vee \mathfrak{M}_{12}] \vee [[\mathfrak{X}_2 \setminus \mathfrak{X}_{12}] \vee \mathfrak{X}_{12}] .$$

Making use of commutative and associative laws (38) rewrite the last equation as follows,

$$[[\mathfrak{M}_1 \setminus \mathfrak{M}_{12}] \vee [\mathfrak{X}_1 \setminus \mathfrak{X}_{12}]] \vee [\mathfrak{X}_{12} \vee \mathfrak{M}_{12}] = [[\mathfrak{M}_2 \setminus \mathfrak{M}_{12}] \vee [\mathfrak{X}_2 \setminus \mathfrak{X}_{12}]] \vee [\mathfrak{X}_{12} \vee \mathfrak{M}_{12}] . \quad (65)$$

According to (46) take a complement of multiset  $\mathfrak{X}_{12} \vee \mathfrak{M}_{12}$  in the l.h.s. and r.h.s. of (65),

$$[\mathfrak{M}_1 \setminus \mathfrak{M}_{12}] \vee [\mathfrak{X}_1 \setminus \mathfrak{X}_{12}] = [\mathfrak{M}_2 \setminus \mathfrak{M}_{12}] \vee [\mathfrak{X}_2 \setminus \mathfrak{X}_{12}] . \quad (66)$$

However, in accordance with the 2nd part of Lemma 5 two following pairs of multisets are disjoint,

$$[\mathfrak{M}_1 \setminus \mathfrak{M}_{12}] \wedge [\mathfrak{M}_2 \setminus \mathfrak{M}_{12}] = \emptyset , \quad [\mathfrak{X}_1 \setminus \mathfrak{X}_{12}] \wedge [\mathfrak{X}_2 \setminus \mathfrak{X}_{12}] = \emptyset . \quad (67)$$

Comparing multiset equality (66) supplied with conditions (67) and Lemma 4 we arrive at (63).  $\square$

## 4.2 Multisets $\mathfrak{M}_1$ , $\mathfrak{M}_2$ and their intersection $\mathfrak{M}_{12}$

In this section we give a detailed description of the multisets  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  which were introduced in section 4.1. The multisets  $\mathfrak{X}_1$  and  $\mathfrak{X}_2$  will be constructed in sections 5 ( $m = 2n$ ) and 6 ( $m = 2n + 1$ ).

Consider the 1st line in Eq. (29) which is the only giving rise to multisets  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$ ,

$$\Lambda(\mathbf{d}^m; z) = \sum_{r=1}^{\gamma(\mathbf{d}^m)} \left[ \sum_{j=1}^m z^{d_j + F_r} - \sum_{j>k=1}^m z^{d_j + d_k + F_r} + \dots + (-1)^m \sum_{j=1}^m z^{\Sigma_m - d_j + F_r} \right] . \quad (68)$$

Making use of definitions (36) and (39) of the operations  $\vee$  and  $\widehat{\bigoplus}$ , construct a sequence of multisets

$$\mathfrak{L}_i(\mathbf{d}^m) = \mathfrak{D}_i(\mathbf{d}^m) \widehat{\bigoplus} \Delta_{\mathcal{H}}(\mathbf{d}^m), \quad i = \leq m - 1 , \quad \text{where} \quad \mathfrak{D}_i(\mathbf{d}^m) = \langle \mathbb{D}_i(\mathbf{d}^m), \sigma_{\mathbb{D}_i} \rangle . \quad (69)$$

The underlying sets  $\mathbb{D}_i(\mathbf{d}^m)$  of degrees  $x_j$  and the mapping functions  $\sigma_{\mathbb{D}_i}(x_j)$  are given by

$$\begin{aligned} \mathbb{D}_1(\mathbf{d}^m) &= \bigcup_{i=1}^m \{d_i\}, \quad \mathbb{D}_2(\mathbf{d}^m) = \bigcup_{i_1>i_2=1}^m \{d_{i_1} + d_{i_2}\}, \quad \mathbb{D}_3(\mathbf{d}^m) = \bigcup_{i_1>i_2>i_3=1}^m \{d_{i_1} + d_{i_2} + d_{i_3}\}, \dots \\ \sigma_{\mathbb{D}_1}(d_i) &= 1, \quad \sigma_{\mathbb{D}_2}(d_{i_1} + d_{i_2}) = 1, \quad \sigma_{\mathbb{D}_3}(d_{i_1} + d_{i_2} + d_{i_3}) = 1, \quad \dots \end{aligned} \quad (70)$$

Due to (70) we have  $\#\mathfrak{D}_i(\mathbf{d}^m) = \binom{m}{i}$  and according to (40) the entire cardinality  $\#\mathfrak{L}_i(\mathbf{d}^m)$  reads,

$$\#\mathfrak{L}_i(\mathbf{d}^m) = \gamma(\mathbf{d}^m) \cdot \#\mathfrak{D}_i(\mathbf{d}^m) , \quad \text{and by definition} \quad \#\mathfrak{L}_0(\mathbf{d}^m) = 0 . \quad (71)$$

Continue to compose multisets and construct two other auxiliary multisets,

$$\begin{aligned}\mathfrak{L}_o(\mathbf{d}^m) &= \mathfrak{L}_1(\mathbf{d}^m) \vee \mathfrak{L}_3(\mathbf{d}^m) \vee \dots \vee \mathfrak{L}_{2\lfloor \frac{m}{2} \rfloor - 1}(\mathbf{d}^m) , & \mathfrak{L}_o(\mathbf{d}^m) &= \langle \mathbb{L}_o(\mathbf{d}^m), \sigma_{\mathbb{L}_o} \rangle , \\ \mathfrak{L}_e(\mathbf{d}^m) &= \mathfrak{L}_2(\mathbf{d}^m) \vee \mathfrak{L}_4(\mathbf{d}^m) \vee \dots \vee \mathfrak{L}_{2\lfloor \frac{m-1}{2} \rfloor}(\mathbf{d}^m) , & \mathfrak{L}_e(\mathbf{d}^m) &= \langle \mathbb{L}_e(\mathbf{d}^m), \sigma_{\mathbb{L}_e} \rangle .\end{aligned}\quad (72)$$

In (72) subscripts 'o' and 'e' stand for the odd and even numbers of summands  $d_j$  in the elements of  $\mathfrak{L}_k(\mathbf{d}^m)$ , respectively. By consequence of distributive property (42) of  $\widehat{\bigoplus}$  over  $\vee$  we have,

$$\begin{aligned}\mathfrak{L}_o(\mathbf{d}^m) &= \mathfrak{D}_o(\mathbf{d}^m) \widehat{\bigoplus} \Delta_{\mathcal{H}}(\mathbf{d}^m) , & \mathfrak{D}_o(\mathbf{d}^m) &= \mathfrak{D}_1(\mathbf{d}^m) \vee \dots \vee \mathfrak{D}_{2\lfloor \frac{m}{2} \rfloor - 1}(\mathbf{d}^m) , \\ \mathfrak{L}_e(\mathbf{d}^m) &= \mathfrak{D}_e(\mathbf{d}^m) \widehat{\bigoplus} \Delta_{\mathcal{H}}(\mathbf{d}^m) , & \mathfrak{D}_e(\mathbf{d}^m) &= \mathfrak{D}_2(\mathbf{d}^m) \vee \dots \vee \mathfrak{D}_{2\lfloor \frac{m-1}{2} \rfloor}(\mathbf{d}^m) .\end{aligned}\quad (73)$$

Represent the 1st line in Eq. (29) (see (68)) in terms of the multisets elements  $\lambda_o$  and  $\lambda_e$ ,

$$\Lambda(\mathbf{d}^m; z) = \sum_{\lambda_o \in \mathbb{L}_o(\mathbf{d}^m)} z^{\lambda_o} - \sum_{\lambda_e \in \mathbb{L}_e(\mathbf{d}^m)} z^{\lambda_e} , \quad (74)$$

where two underlying sets  $\mathbb{L}_o(\mathbf{d}^m)$  and  $\mathbb{L}_e(\mathbf{d}^m)$  were introduced in (72).

Keeping in mind the definition of the multisets  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  given in section 4.1 (after Eq. (60)) and comparing the difference  $\Lambda(\mathbf{d}^m; z)$  in (74) coming from (68) with recasted Eq. (60) we conclude that  $\mathfrak{L}_o(\mathbf{d}^m)$ ,  $\mathfrak{L}_e(\mathbf{d}^m)$  and  $\mathfrak{L}_e(\mathbf{d}^m) \wedge \mathfrak{L}_o(\mathbf{d}^m)$  are exactly the multisets  $\mathfrak{M}_1$ ,  $\mathfrak{M}_2$  and  $\mathfrak{M}_{12}$  appeared in (61) and Theorem 4 and equipped with dependence on  $\mathbf{d}^m$ ,

$$\mathfrak{M}_1(\mathbf{d}^m) \equiv \mathfrak{L}_o(\mathbf{d}^m) , \quad \mathfrak{M}_2(\mathbf{d}^m) \equiv \mathfrak{L}_e(\mathbf{d}^m) , \quad \mathfrak{M}_{12}(\mathbf{d}^m) \equiv \mathfrak{L}_e(\mathbf{d}^m) \wedge \mathfrak{L}_o(\mathbf{d}^m) . \quad (75)$$

### 4.3 Multiset $\mathfrak{D}_e(\mathbf{d}^m) \wedge \mathfrak{D}_o(\mathbf{d}^m)$ for small edim

In the case of pseudosymmetric semigroup (see section 2.1) the expressions for  $\mathfrak{M}_{12}(\mathbf{d}^m)$  and its cardinality are simplified considerably. Indeed, since  $\#\Delta_{\mathcal{H}}(\mathbf{d}^m) = 1$  then by (73) we get

$$\mathfrak{L}_o(\mathbf{d}^m) = \mathfrak{D}_o(\mathbf{d}^m) \oplus \{F(\mathbf{d}^m)/2\} , \quad \mathfrak{L}_e(\mathbf{d}^m) = \mathfrak{D}_e(\mathbf{d}^m) \oplus \{F(\mathbf{d}^m)/2\} .$$

Define a new multiset,  $\mathfrak{D}_{eo}(\mathbf{d}^m) = \mathfrak{D}_e(\mathbf{d}^m) \wedge \mathfrak{D}_o(\mathbf{d}^m)$  and represent  $\mathfrak{M}_{12}(\mathbf{d}^m)$  according to (75)

$$\mathfrak{M}_{12}(\mathbf{d}^m) = \mathfrak{D}_{eo}(\mathbf{d}^m) \oplus \{1/2 F(\mathbf{d}^m)\} , \quad \text{and} \quad \#\mathfrak{M}_{12}(\mathbf{d}^m) = \#\mathfrak{D}_{eo}(\mathbf{d}^m) .$$

In the sequel we address the following questions: how big can be the cardinality  $\#\mathfrak{D}_{eo}(\mathbf{d}^m)$  and when it does vanish. For generic tuple  $\mathbf{d}^m$  these questions are addressed to the additive number theory. Here we give answer for small edim,  $m = 3, 4, 5$ , and return to arbitrary edim elsewhere.

**Remark 1** Making use of representation (37) for multisets  $\mathfrak{D}_k(\mathbf{d}^m)$  and unique occurrence (70) of any sum  $\omega_k = d_{i_1} + d_{i_2} + \dots + d_{i_k}$ ,  $1 \leq k < m$ , of generators  $d_j$  therein we skip (in this section) a unity  $\sigma_{\mathbb{D}_k}(\omega_k) = 1$  in representation (37). E.g. we shall write,

$$\mathfrak{D}_2(\mathbf{d}^4) = \{d_1 + d_2, d_1 + d_3, d_1 + d_4, d_2 + d_3, d_2 + d_4, d_3 + d_4\}.$$

**Proposition 2** *Two multisets  $\mathfrak{D}_{eo}(\mathbf{d}^3)$  and  $\mathfrak{D}_{eo}(\mathbf{d}^4)$  are empty.*

**Proof** First, according to (73) write  $\mathfrak{D}_o(\mathbf{d}^3) = \mathfrak{D}_1(\mathbf{d}^3)$  and  $\mathfrak{D}_e(\mathbf{d}^3) = \mathfrak{D}_2(\mathbf{d}^3)$ , so

$$\mathfrak{D}_o(\mathbf{d}^3) = \{d_1, d_2, d_3\} , \quad \mathfrak{D}_e(\mathbf{d}^3) = \{d_1 + d_2, d_2 + d_3, d_3 + d_1\} .$$

Thus,  $\mathfrak{D}_{eo}(\mathbf{d}^3) = \emptyset$ , otherwise the minimality of the generating set  $\mathbf{d}^3$  would be broken. Next, according to (73) write  $\mathfrak{D}_o(\mathbf{d}^4) = \mathfrak{D}_1(\mathbf{d}^4) \vee \mathfrak{D}_3(\mathbf{d}^4)$  and  $\mathfrak{D}_e(\mathbf{d}^4) = \mathfrak{D}_2(\mathbf{d}^4)$ , so

$$\begin{aligned} \mathfrak{D}_o(\mathbf{d}^4) &= \{d_1, d_2, d_3, d_4, d_1 + d_2 + d_3, d_2 + d_3 + d_4, d_3 + d_4 + d_1, d_1 + d_3 + d_4\} , \\ \mathfrak{D}_e(\mathbf{d}^4) &= \{d_1 + d_2, d_1 + d_3, d_1 + d_4, d_2 + d_3, d_2 + d_4, d_3 + d_4\} . \end{aligned}$$

Thus,  $\mathfrak{D}_{eo}(\mathbf{d}^4) = \emptyset$  by the same reason of minimality of the generating set  $\mathbf{d}^4$ .  $\square$

**Proposition 3** *Let a numerical semigroup  $S(\mathbf{d}^5)$  be given. Then  $\#\mathfrak{D}_{eo}(\mathbf{d}^5) \leq 1$ .*

**Proof** Let a tuple  $\mathbf{d}^5$  be given, then according to (73) write

$$\mathfrak{D}_o(\mathbf{d}^5) = \mathfrak{D}_1(\mathbf{d}^5) \vee \mathfrak{D}_3(\mathbf{d}^5) , \quad \mathfrak{D}_e(\mathbf{d}^5) = \mathfrak{D}_2(\mathbf{d}^5) \vee \mathfrak{D}_4(\mathbf{d}^5) .$$

It is easy to verify that

$$\mathfrak{D}_1(\mathbf{d}^5) \wedge \mathfrak{D}_2(\mathbf{d}^5) = \mathfrak{D}_1(\mathbf{d}^5) \wedge \mathfrak{D}_4(\mathbf{d}^5) = \mathfrak{D}_3(\mathbf{d}^5) \wedge \mathfrak{D}_4(\mathbf{d}^5) = \emptyset ,$$

by the reason of minimality of the generating set  $\mathbf{d}^5$ . There is left a multiset  $\mathfrak{D}_2(\mathbf{d}^5) \wedge \mathfrak{D}_3(\mathbf{d}^5)$  which we'll study. Write all  $\binom{5}{2} \binom{5-2}{3} = 10$  admissible relations and check their compatibility,

$$\begin{aligned} &1) \ d_1 + d_2 = d_3 + d_4 + d_5 , \quad 2) \ d_1 + d_3 = d_2 + d_4 + d_5 , \quad 3) \ d_1 + d_4 = d_2 + d_3 + d_5 , \\ &4) \ d_1 + d_5 = d_2 + d_3 + d_4 , \quad 5) \ d_2 + d_3 = d_1 + d_4 + d_5 , \quad 6) \ d_2 + d_4 = d_1 + d_3 + d_5 , \\ &7) \ d_2 + d_5 = d_1 + d_3 + d_4 , \quad 8) \ d_3 + d_4 = d_1 + d_2 + d_5 , \quad 9) \ d_3 + d_5 = d_1 + d_2 + d_4 , \\ &10) \ d_4 + d_5 = d_1 + d_2 + d_3 . \end{aligned} \tag{76}$$

Since the ordering in the set  $\{1, 2, 3, 4, 5\}$  is arbitrary it is sufficient to check the compatibility of the 1st equality in (76) with the other nine. By inspection of compatibility of all 9 pairs we conclude that it contradicts the minimality of the generating set  $\{d_1, d_2, d_3, d_4, d_5\}$ , e. g.

$$\left\{ \begin{array}{l} 1) \ d_1 + d_2 = d_3 + d_4 + d_5 \\ 2) \ d_1 + d_3 = d_2 + d_4 + d_5 \end{array} \right. \rightarrow d_2 = d_3 , \quad \left\{ \begin{array}{l} 1) \ d_1 + d_2 = d_3 + d_4 + d_5 \\ 9) \ d_3 + d_5 = d_1 + d_2 + d_4 \end{array} \right. \rightarrow d_5 = 0 , \quad \text{etc} .$$

Thus, there exists at most one admissible relation and Proposition is proven.  $\square$



## 5 Almost Symmetric Semigroups $S(\mathbf{d}^{2n})$ , $n \geq 2$

In (61) we have defined a multiset equality associated with Eq. (29) and based on two multisets  $\mathfrak{M}_1, \mathfrak{M}_2$  of given gaps  $F_j$  and generators  $d_i$ , and two multisets  $\mathfrak{X}_1, \mathfrak{X}_2$  of degrees  $C_{j,i}$  and  $\overline{C}_{j,i}$ . The first two multisets  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  were constructed explicitly in (75). In this section we construct the other two multisets  $\mathfrak{X}_1$  and  $\mathfrak{X}_2$  providing their consistence with Eqs. (29) and (61).

An interchange of signs of the terms in Eq. (29) and factor  $(-1)^{m-1}$  make our analysis not easy, this can be seen for edim of distinct parities,  $m = 2n$  and  $m = 2n + 1$ , where the multisets  $\mathfrak{X}_1$  and  $\mathfrak{X}_2$  are composed in different ways. Therefore we consider two cases of even and odd edim separately, and start with for  $m = 2n$ . Substituting into the 1st line of Eq. (29) its representation given in (74) and (75), write the whole Eq. (29) in the form which is similar to (60),

$$\sum_{\omega_1 \in \mathbb{M}_1(\mathbf{d}^{2n})} z^{\omega_1} + \sum_{q=1}^{n-1} \sum_{j=1}^{\beta_{2q-1}(\mathbf{d}^{2n})} \left[ z^{C_{j,2q-1}} + z^{\overline{C}_{j,2q-1}} \right] = \sum_{\omega_2 \in \mathbb{M}_2(\mathbf{d}^{2n})} z^{\omega_2} + \sum_{q=1}^{n-1} \sum_{j=1}^{\beta_{2q}(\mathbf{d}^{2n})} \left[ z^{C_{j,2q}} + z^{\overline{C}_{j,2q}} \right] \quad (77)$$

By comparison of Eqs. (77) and (60) we'll find the multisets  $\mathfrak{X}_1(\mathbf{d}^{2n})$  and  $\mathfrak{X}_2(\mathbf{d}^{2n})$ .

Consider the last sums in the l.h.s. and r.h.s. of Eq. (77) and construct two auxiliary multisets  $\mathfrak{B}_i(\mathbf{d}^{2n})$  and  $\overline{\mathfrak{B}}_i(\mathbf{d}^{2n})$  of syzygies degrees which have a standard representations (see section 4) through the sets  $\mathbb{B}_i(\mathbf{d}^{2n})$  and  $\overline{\mathbb{B}}_i(\mathbf{d}^{2n})$  defined in (7) and (30),

$$\mathfrak{B}_i(\mathbf{d}^{2n}) = \langle \mathbb{B}_i(\mathbf{d}^{2n}), \sigma_{\mathbb{B}_i(\mathbf{d}^{2n})} \rangle, \quad \overline{\mathfrak{B}}_i(\mathbf{d}^{2n}) = \langle \overline{\mathbb{B}}_i(\mathbf{d}^{2n}), \sigma_{\overline{\mathbb{B}}_i(\mathbf{d}^{2n})} \rangle. \quad (78)$$

By definitions (7) and (30) of the sets  $\mathbb{B}_i(\mathbf{d}^{2n})$  and  $\overline{\mathbb{B}}_i(\mathbf{d}^{2n})$  we have  $\sigma_{\mathbb{B}_i(\mathbf{d}^{2n})}(C_{j,i}) = \sigma_{\overline{\mathbb{B}}_i(\mathbf{d}^{2n})}(\overline{C}_{j,i})$  that together with (78) leads to  $\#\mathfrak{B}_i(\mathbf{d}^{2n}) = \#\overline{\mathfrak{B}}_i(\mathbf{d}^{2n}) = \beta_i(\mathbf{d}^{2n})$ . Define the following multisets,

$$\begin{aligned} \mathfrak{B}_o(\mathbf{d}^{2n}) &= \bigvee_{i=1}^{n-1} \mathfrak{B}_{2i-1}(\mathbf{d}^{2n}), & \mathfrak{B}_e(\mathbf{d}^{2n}) &= \bigvee_{i=1}^{n-1} \mathfrak{B}_{2i}(\mathbf{d}^{2n}), \\ \overline{\mathfrak{B}}_o(\mathbf{d}^{2n}) &= \bigvee_{i=1}^{n-1} \overline{\mathfrak{B}}_{2i-1}(\mathbf{d}^{2n}), & \overline{\mathfrak{B}}_e(\mathbf{d}^{2n}) &= \bigvee_{i=1}^{n-1} \overline{\mathfrak{B}}_{2i}(\mathbf{d}^{2n}), \end{aligned} \quad (79)$$

where subscripts 'o' and 'e' stand for the odd  $q = 2i - 1$  and even  $q = 2i$  indices, respectively, of summands  $\mathfrak{B}_q(\mathbf{d}^{2n})$  and  $\overline{\mathfrak{B}}_q(\mathbf{d}^{2n})$ .

By comparison of Eqs. (77) and (60) we can define the multisets  $\mathfrak{X}_1(\mathbf{d}^{2n})$ ,  $\mathfrak{X}_2(\mathbf{d}^{2n})$  and their intersection  $\mathfrak{X}_{12}(\mathbf{d}^{2n})$  through four multisets (79) and two multiset operations  $\bigvee$  and  $\bigwedge$ ,

$$\mathfrak{X}_1(\mathbf{d}^{2n}) \equiv \mathfrak{B}_o(\mathbf{d}^{2n}) \bigvee \overline{\mathfrak{B}}_o(\mathbf{d}^{2n}), \quad \mathfrak{X}_2(\mathbf{d}^{2n}) \equiv \mathfrak{B}_e(\mathbf{d}^{2n}) \bigvee \overline{\mathfrak{B}}_e(\mathbf{d}^{2n}), \quad (80)$$

$$\mathfrak{X}_{12}(\mathbf{d}^{2n}) \equiv \left[ \mathfrak{B}_o(\mathbf{d}^{2n}) \bigvee \overline{\mathfrak{B}}_o(\mathbf{d}^{2n}) \right] \bigwedge \left[ \mathfrak{B}_e(\mathbf{d}^{2n}) \bigvee \overline{\mathfrak{B}}_e(\mathbf{d}^{2n}) \right]. \quad (81)$$

Substituting (75) and (80) into multiset equality (61) we get

$$\mathfrak{L}_o(\mathbf{d}^{2n}) \vee \mathfrak{B}_o(\mathbf{d}^{2n}) \vee \overline{\mathfrak{B}}_o(\mathbf{d}^{2n}) = \mathfrak{L}_e(\mathbf{d}^{2n}) \vee \mathfrak{B}_e(\mathbf{d}^{2n}) \vee \overline{\mathfrak{B}}_e(\mathbf{d}^{2n}) .$$

**Lemma 6** *Let an almost symmetric semigroup  $S(\mathbf{d}^{2n})$  be given. Then*

$$\begin{aligned} [\mathfrak{B}_o(\mathbf{d}^{2n}) \vee \overline{\mathfrak{B}}_o(\mathbf{d}^{2n})] \setminus \mathfrak{X}_{12}(\mathbf{d}^{2n}) &= \mathfrak{L}_e(\mathbf{d}^{2n}) \setminus \mathfrak{M}_{12}(\mathbf{d}^{2n}) , \\ [\mathfrak{B}_e(\mathbf{d}^{2n}) \vee \overline{\mathfrak{B}}_e(\mathbf{d}^{2n})] \setminus \mathfrak{X}_{12}(\mathbf{d}^{2n}) &= \mathfrak{L}_o(\mathbf{d}^{2n}) \setminus \mathfrak{M}_{12}(\mathbf{d}^{2n}) . \end{aligned} \quad (82)$$

**Proof** Substituting the expressions (75) for multisets  $\mathfrak{M}_i(\mathbf{d}^{2n})$ ,  $i = 1, 2$ , and expressions (80) for multisets  $\mathfrak{X}_i(\mathbf{d}^{2n})$ ,  $i = 1, 2$ , into equality (61) we apply Theorem 4. Thus, by consequence of (63) we arrive at (82).  $\square$

Lemma 6 does not give yet explicit expressions for syzygies degrees  $C_{j,i}$  and  $\overline{C}_{j,i}$  since it is hard to differentiate them one from another. This requires much more powerful algebraic methods, e.g. the Hilbert syzygy theorem [2]. However, Lemma 6 leads to new relations for the Betti numbers. Define the following cardinalities:  $\#\mathfrak{M}_{12}(\mathbf{d}^m) = \ell(\mathbf{d}^m)$ ,  $\#\mathfrak{X}_{12}(\mathbf{d}^m) = \wp(\mathbf{d}^m)$  and  $\delta(\mathbf{d}^m) = \wp(\mathbf{d}^m) - \ell(\mathbf{d}^m)$  and prove Theorem.

**Theorem 5** *Let an almost symmetric semigroup  $S(\mathbf{d}^{2n})$  be given. Then*

$$\begin{aligned} \beta_1(\mathbf{d}^{2n}) + \beta_3(\mathbf{d}^{2n}) + \dots + \beta_{2n-3}(\mathbf{d}^{2n}) &= \gamma(\mathbf{d}^{2n}) \cdot (4^{n-1} - 1) + \frac{1}{2} \delta(\mathbf{d}^{2n}) , \\ \beta_2(\mathbf{d}^{2n}) + \beta_4(\mathbf{d}^{2n}) + \dots + \beta_{2n-2}(\mathbf{d}^{2n}) &= \gamma(\mathbf{d}^{2n}) \cdot 4^{n-1} + \frac{1}{2} \delta(\mathbf{d}^{2n}) . \end{aligned} \quad (83)$$

**Proof** By Lemma 6 and in view of definition (36) of the operation  $\vee$  we get

$$\begin{aligned} \#\mathfrak{B}_o(\mathbf{d}^{2n}) + \#\overline{\mathfrak{B}}_o(\mathbf{d}^{2n}) - \wp(\mathbf{d}^{2n}) &= \#\mathfrak{L}_e(\mathbf{d}^{2n}) - \ell(\mathbf{d}^{2n}) , \\ \#\mathfrak{B}_e(\mathbf{d}^{2n}) + \#\overline{\mathfrak{B}}_e(\mathbf{d}^{2n}) - \wp(\mathbf{d}^{2n}) &= \#\mathfrak{L}_o(\mathbf{d}^{2n}) - \ell(\mathbf{d}^{2n}) . \end{aligned}$$

Making use of (69), (72) and (78), (79), and inserting them into the last equations we arrive at

$$\begin{aligned} \beta_1(\mathbf{d}^{2n}) + \beta_3(\mathbf{d}^{2n}) + \dots + \beta_{2n-3}(\mathbf{d}^{2n}) &= \frac{\gamma(\mathbf{d}^{2n})}{2} \left[ \binom{2n}{2} + \binom{2n}{4} + \dots + \binom{2n}{2n-2} \right] + \frac{\delta(\mathbf{d}^{2n})}{2} , \\ \beta_2(\mathbf{d}^{2n}) + \beta_4(\mathbf{d}^{2n}) + \dots + \beta_{2n-2}(\mathbf{d}^{2n}) &= \frac{\gamma(\mathbf{d}^{2n})}{2} \left[ \binom{2n}{1} + \binom{2n}{3} + \dots + \binom{2n}{2n-1} \right] + \frac{\delta(\mathbf{d}^{2n})}{2} . \end{aligned}$$

A simple algebraic exercise gives,

$$\binom{2n}{2} + \binom{2n}{4} + \dots + \binom{2n}{2n-2} = 2^{2n-1} - 2 , \quad \binom{2n}{1} + \binom{2n}{3} + \dots + \binom{2n}{2n-1} = 2^{2n-1} ,$$

that bring us to (83).  $\square$

By consequence of (83) and the fact that the Betti numbers are nonnegative integers it follows that  $\wp(\mathbf{d}^{2n}) = \ell(\mathbf{d}^{2n}) \pmod{2}$ .

The case of pseudosymmetric semigroup,  $\gamma(\mathbf{d}^m) = 1$ , of embedding dimension 4 is most simple. By Proposition 2 and Theorem 5 we have here,

$$\beta_1(\mathbf{d}^4) = 3 + \frac{1}{2} \wp(\mathbf{d}^4) , \quad \beta_2(\mathbf{d}^4) = 4 + \frac{1}{2} \wp(\mathbf{d}^4) . \quad (84)$$

**Example 1**  $\{d_1, d_2, d_3, d_4\} = \{5, 6, 7, 9\}$  ,  $\beta_1 = 5$  ,  $\beta_2 = 6$  ,  $\beta_3 = 2$  ,

$$B_3(5, 6, 7, 9) = \{31, 35\} , \quad S'(5, 6, 7, 9) = \{4, 8\} , \quad \Delta_{\mathcal{H}}(5, 6, 7, 9) = \{4\} , \quad \sum_4 = 27 ,$$

$$\Delta_{\mathcal{G}}(5, 6, 7, 9) = \{1, 2, 3, 8\} , \quad F(5, 6, 7, 9) = 8 , \quad \ell(5, 6, 7, 9) = 0 , \quad \wp(5, 6, 7, 9) = 4 ,$$

$$Q(5, 6, 7, 9; z) = 1 - z^{12} - z^{14} - z^{15} - z^{16} - z^{18} + z^{21} + z^{22} + z^{23} + z^{24} + z^{25} + z^{26} - z^{31} - z^{35} .$$

## 6 Almost Symmetric Semigroups $S(\mathbf{d}^{2n+1})$ , $n \geq 1$

Substituting into the 1st line of Eq. (29) its representation given in (74) and (75), write Eq. (29) for  $m = 2n + 1$  as follows,

$$\begin{aligned} \sum_{\omega_1 \in \mathbb{M}_1(\mathbf{d}^{2n+1})} z^{\omega_1} + \sum_{q=1}^n \sum_{j=1}^{\beta_{2q-1}(\mathbf{d}^{2n+1})} z^{C_{j,2q-1}} + \sum_{q=1}^{n-1} \sum_{j=1}^{\beta_{2q}(\mathbf{d}^{2n+1})} z^{\overline{C}_{j,2q}} = \\ \sum_{\omega_2 \in \mathbb{M}_2(\mathbf{d}^{2n+1})} z^{\omega_2} + \sum_{q=1}^{n-1} \sum_{j=1}^{\beta_{2q}(\mathbf{d}^{2n+1})} z^{C_{j,2q}} + \sum_{q=1}^n \sum_{j=1}^{\beta_{2q-1}(\mathbf{d}^{2n+1})} z^{\overline{C}_{j,2q-1}} . \end{aligned} \quad (85)$$

By comparison of Eqs. (85) and (60) we'll find the multisets  $\mathfrak{X}_1(\mathbf{d}^{2n+1})$  and  $\mathfrak{X}_2(\mathbf{d}^{2n+1})$ .

Consider the last sums in the l.h.s. and r.h.s. of Eq. (85) and construct two auxiliary multisets  $\mathfrak{B}_i(\mathbf{d}^{2n+1})$  and  $\overline{\mathfrak{B}}_i(\mathbf{d}^{2n+1})$  of syzygies degrees which have a standard representations (see section 4) through the sets  $\mathbb{B}_i(\mathbf{d}^{2n+1})$  and  $\overline{\mathbb{B}}_i(\mathbf{d}^{2n+1})$  defined in (7) and (30),

$$\mathfrak{B}_i(\mathbf{d}^{2n+1}) = \langle \mathbb{B}_i(\mathbf{d}^{2n+1}), \sigma_{\mathbb{B}_i(\mathbf{d}^{2n+1})} \rangle , \quad \overline{\mathfrak{B}}_i(\mathbf{d}^{2n+1}) = \langle \overline{\mathbb{B}}_i(\mathbf{d}^{2n+1}), \sigma_{\overline{\mathbb{B}}_i(\mathbf{d}^{2n+1})} \rangle . \quad (86)$$

By definitions (7) and (30) of the underlying sets we have  $\sigma_{\mathbb{B}_i(\mathbf{d}^{2n+1})}(C_{j,i}) = \sigma_{\overline{\mathbb{B}}_i(\mathbf{d}^{2n+1})}(\overline{C}_{j,i})$  that together with (86) leads to  $\#\mathfrak{B}_i(\mathbf{d}^{2n+1}) = \#\overline{\mathfrak{B}}_i(\mathbf{d}^{2n+1}) = \beta_i(\mathbf{d}^{2n+1})$ . Define four other multisets,

$$\begin{aligned} \mathfrak{B}_o(\mathbf{d}^{2n+1}) &= \bigvee_{i=1}^n \mathfrak{B}_{2i-1}(\mathbf{d}^{2n+1}) , \quad \mathfrak{B}_e(\mathbf{d}^{2n+1}) = \bigvee_{i=1}^{n-1} \mathfrak{B}_{2i}(\mathbf{d}^{2n+1}) , \\ \overline{\mathfrak{B}}_o(\mathbf{d}^{2n+1}) &= \bigvee_{i=1}^n \overline{\mathfrak{B}}_{2i-1}(\mathbf{d}^{2n+1}) , \quad \overline{\mathfrak{B}}_e(\mathbf{d}^{2n+1}) = \bigvee_{i=1}^{n-1} \overline{\mathfrak{B}}_{2i}(\mathbf{d}^{2n+1}) , \end{aligned} \quad (87)$$

where subscripts 'o' and 'e' stand for the odd  $q = 2i - 1$  and even  $q = 2i$  indices, respectively, of summands  $\mathfrak{B}_q(\mathbf{d}^{2n+1})$  and  $\overline{\mathfrak{B}}_q(\mathbf{d}^{2n+1})$ .

By comparison of Eqs. (85) and (60) we can define the multisets  $\mathfrak{X}_1(\mathbf{d}^{2n+1})$ ,  $\mathfrak{X}_2(\mathbf{d}^{2n+1})$  and their intersection  $\mathfrak{X}_{12}(\mathbf{d}^{2n+1})$  through four multisets (87) and two multiset operations  $\bigvee$  and  $\bigwedge$ ,

$$\mathfrak{X}_1(\mathbf{d}^{2n+1}) \equiv \mathfrak{B}_o(\mathbf{d}^{2n+1}) \bigvee \overline{\mathfrak{B}}_e(\mathbf{d}^{2n+1}), \quad \mathfrak{X}_2(\mathbf{d}^{2n+1}) \equiv \mathfrak{B}_e(\mathbf{d}^{2n+1}) \bigvee \overline{\mathfrak{B}}_o(\mathbf{d}^{2n+1}), \quad (88)$$

$$\mathfrak{X}_{12}(\mathbf{d}^{2n+1}) \equiv [\mathfrak{B}_o(\mathbf{d}^{2n+1}) \bigvee \overline{\mathfrak{B}}_e(\mathbf{d}^{2n+1})] \bigwedge [\mathfrak{B}_e(\mathbf{d}^{2n+1}) \bigvee \overline{\mathfrak{B}}_o(\mathbf{d}^{2n+1})]. \quad (89)$$

Substituting (75) and (88) into multiset equality (61) we get

$$\mathfrak{L}_o(\mathbf{d}^{2n+1}) \bigvee \mathfrak{B}_o(\mathbf{d}^{2n+1}) \bigvee \overline{\mathfrak{B}}_e(\mathbf{d}^{2n+1}) = \mathfrak{L}_e(\mathbf{d}^{2n+1}) \bigvee \mathfrak{B}_e(\mathbf{d}^{2n+1}) \bigvee \overline{\mathfrak{B}}_o(\mathbf{d}^{2n+1}).$$

**Lemma 7** *Let an almost symmetric semigroup  $S(\mathbf{d}^{2n+1})$  be given. Then*

$$\begin{aligned} [\mathfrak{B}_o(\mathbf{d}^{2n+1}) \bigvee \overline{\mathfrak{B}}_e(\mathbf{d}^{2n+1})] \setminus \mathfrak{X}_{12}(\mathbf{d}^{2n+1}) &= \mathfrak{L}_e(\mathbf{d}^{2n+1}) \setminus \mathfrak{M}_{12}(\mathbf{d}^{2n+1}), \\ [\mathfrak{B}_e(\mathbf{d}^{2n+1}) \bigvee \overline{\mathfrak{B}}_o(\mathbf{d}^{2n+1})] \setminus \mathfrak{X}_{12}(\mathbf{d}^{2n+1}) &= \mathfrak{L}_o(\mathbf{d}^{2n+1}) \setminus \mathfrak{M}_{12}(\mathbf{d}^{2n+1}). \end{aligned} \quad (90)$$

**Proof** Substituting the expressions (75) for multisets  $\mathfrak{M}_i(\mathbf{d}^{2n+1})$ ,  $i = 1, 2$ , and expressions (88) for multisets  $\mathfrak{X}_i(\mathbf{d}^{2n+1})$ ,  $i = 1, 2$ , into equality (61) we apply Theorem 4. Thus, by consequence of (63) we arrive at (90).  $\square$

**Theorem 6** *Let an almost symmetric semigroup  $S(\mathbf{d}^{2n+1})$  be given. Then*

$$\begin{aligned} \beta_1(\mathbf{d}^{2n+1}) + \beta_3(\mathbf{d}^{2n+1}) + \dots + \beta_{2n-1}(\mathbf{d}^{2n+1}) &= \gamma(\mathbf{d}^{2n+1}) \cdot 2^{2n-1} + \frac{1}{2} \delta(\mathbf{d}^{2n+1}) + 1, \\ \beta_2(\mathbf{d}^{2n+1}) + \beta_4(\mathbf{d}^{2n+1}) + \dots + \beta_{2n-2}(\mathbf{d}^{2n+1}) &= \gamma(\mathbf{d}^{2n+1}) \cdot (2^{2n-1} - 1) + \frac{1}{2} \delta(\mathbf{d}^{2n+1}) - 1. \end{aligned} \quad (91)$$

**Proof** By Lemma 7 and definition (36) of the operation  $\bigvee$  we have

$$\begin{aligned} \#\mathfrak{B}_o(\mathbf{d}^{2n+1}) + \#\overline{\mathfrak{B}}_e(\mathbf{d}^{2n+1}) - \wp(\mathbf{d}^{2n+1}) &= \#\mathfrak{L}_e(\mathbf{d}^{2n+1}) - \ell(\mathbf{d}^{2n+1}), \\ \#\mathfrak{B}_e(\mathbf{d}^{2n+1}) + \#\overline{\mathfrak{B}}_o(\mathbf{d}^{2n+1}) - \wp(\mathbf{d}^{2n+1}) &= \#\mathfrak{L}_o(\mathbf{d}^{2n+1}) - \ell(\mathbf{d}^{2n+1}). \end{aligned}$$

Making use of (69), (72) and (86), (87), and inserting them into the last equations we get

$$\beta_1(\mathbf{d}^{2n+1}) + \beta_2(\mathbf{d}^{2n+1}) + \dots + \beta_{2n-1}(\mathbf{d}^{2n+1}) = \gamma(\mathbf{d}^{2n+1}) \cdot (4^n - 1) + \delta(\mathbf{d}^{2n+1}).$$

Making sum of the last equality with (8) and simplifying the result we arrive at (91).  $\square$

The following Example of almost symmetric semigroups  $S(\mathbf{d}^5)$  is taken from [1]. We have calculated the Hilbert series, the Betti numbers and the other characteristics.

**Example 2**  $\{d_1, d_2, d_3, d_4, d_5\} = \{6, 7, 8, 10, 11\}$  ,  $\beta_1 = 9$  ,  $\beta_2 = 17$  ,  $\beta_3 = 12$  ,  $\beta_4 = 3$

$$\begin{aligned} B_4(6, 7, 8, 10, 11) &= \{46, 47, 51\} , \quad S'(6, 7, 8, 10, 11) = \{4, 5, 9\} , \quad \Delta_{\mathcal{H}}(6, 7, 8, 10, 11) = \{4, 5\} , \quad \sum_5 = 42 \\ \Delta_{\mathcal{G}}(6, 7, 8, 10, 11) &= \{1, 2, 3, 9\} , \quad F(6, 7, 8, 10, 11) = 9 , \quad \ell(6, 7, 8, 10, 11) = 2 , \quad \wp(6, 7, 8, 10, 11) = 10 , \\ Q(6, 7, 8, 10, 11; z) &= 1 - z^{14} - z^{16} - z^{17} - 2z^{18} - z^{19} - z^{20} - z^{21} - z^{22} + z^{24} + 2z^{25} + 2z^{26} + 2z^{27} + 3z^{28} + \\ &\quad 3z^{29} + 2z^{30} + z^{31} + z^{32} - 2z^{35} - 2z^{36} - 2z^{37} - z^{38} - 2z^{39} - 2z^{40} - z^{41} + z^{46} + z^{47} + z^{51} \end{aligned}$$

**Corollary 2** *Let an almost symmetric semigroup  $S(\mathbf{d}^m)$  be given. Then*

$$\delta(\mathbf{d}^m) \leq [d_1 - \gamma(\mathbf{d}^m)] 2^{m-1} - 2m . \quad (92)$$

**Proof** We prove Corollary for even and odd edim separately. First, consider an almost symmetric semigroup  $S(\mathbf{d}^{2n})$  and calculate the sum of the Betti numbers  $\beta_k(\mathbf{d}^{2n})$ . Keeping in mind  $\beta_{2n-1}(\mathbf{d}^{2n}) = 2n - 1$  and making use of Theorem 5 we get,

$$\sum_{k=0}^{2n-1} \beta_k(\mathbf{d}^{2n}) = \gamma(\mathbf{d}^{2n}) 2^{2n-1} + 2 + \delta(\mathbf{d}^{2n}) . \quad (93)$$

By comparison (93) with (9) we obtain,

$$\delta(\mathbf{d}^{2n}) \leq [d_1 - \gamma(\mathbf{d}^{2n})] 2^{2n-1} - 4n . \quad (94)$$

Next, consider an almost symmetric semigroup  $S(\mathbf{d}^{2n+1})$  and make similar calculations with help of Theorem 6,

$$\sum_{k=0}^{2n} \beta_k(\mathbf{d}^{2n+1}) = \gamma(\mathbf{d}^{2n+1}) 4^n + 2 + \delta(\mathbf{d}^{2n+1}) . \quad (95)$$

By comparison (95) with (9) we obtain,

$$\delta(\mathbf{d}^{2n+1}) \leq [d_1 - \gamma(\mathbf{d}^{2n+1})] 2^{2n} - 2(2n + 1) . \quad (96)$$

Combining formulas (94) and (96) we come to (92).  $\square$

## 6.1 Pseudosymmetric semigroup $S(\mathbf{d}^3)$

This case is mostly simple and makes it possible to find all syzygy degrees and the Frobenius number as well. Keeping in mind  $\beta_1(\mathbf{d}^3) = 3$ ,  $t(\mathbf{d}^3) = 2$ ,  $F_1 = \frac{1}{2}F(\mathbf{d}^3)$  and denoting  $C_{j,1} = e_j$ , write Eq. (85)

$$\sum_{j=1}^3 z^{d_j + \frac{1}{2}F(\mathbf{d}^3)} + \sum_{j=1}^3 z^{e_j} = \sum_{j>k=1}^3 z^{d_j + d_k + \frac{1}{2}F(\mathbf{d}^3)} + \sum_{j=1}^3 z^{-e_j + F(\mathbf{d}^3) + \Sigma_3} .$$

However, by Proposition 2 we have  $\ell(\mathbf{d}^3) = 0$ , and therefore by the 1st equation in (91) it follows

$$3 = 3 + \frac{1}{2}\wp(\mathbf{d}^3) \longrightarrow \wp(\mathbf{d}^3) = 0 \longrightarrow \mathfrak{x}_{12}(\mathbf{d}^3) = \emptyset. \quad (97)$$

By (97) and Lemma 7 the multiset equalities (90) read:  $\mathfrak{B}_1(\mathbf{d}^3) = \mathfrak{L}_2(\mathbf{d}^3)$  and  $\overline{\mathfrak{B}}_1(\mathbf{d}^3) = \mathfrak{L}_1(\mathbf{d}^3)$  that gives two following correspondences,

$$\{e_i\} = \left\{d_j + d_k + \frac{1}{2}F(\mathbf{d}^3)\right\} \quad \text{and} \quad \left\{d_i + \frac{1}{2}F(\mathbf{d}^3)\right\} = \{-e_i + F(\mathbf{d}^3) + \Sigma_3\}, \quad (98)$$

which are consistent each other. Hence, the whole numerator  $Q(\mathbf{d}^3; z)$  in the Hilbert series reads,

$$Q(\mathbf{d}^3; z) = 1 - z^{d_1+d_2+\frac{1}{2}F(\mathbf{d}^3)} - z^{d_2+d_3+\frac{1}{2}F(\mathbf{d}^3)} - z^{d_3+d_1+\frac{1}{2}F(\mathbf{d}^3)} + z^{\frac{1}{2}F(\mathbf{d}^3)+\Sigma_3} + z^{F(\mathbf{d}^3)+\Sigma_3}.$$

The last expression makes it possible to derive the explicit formulas for the Frobenius number  $F(\mathbf{d}^3)$  and genus  $G(\mathbf{d}^3)$  of the 3D pseudosymmetric semigroups. For this purpose we'll make use of formulas for generic 3D nonsymmetric semigroups which were established in [3], Ch. 6,

$$2F(\mathbf{d}^3) = E_1 + \sqrt{E_1^2 - 4E_2 + 4D_3} - 2D_1, \quad 2G(\mathbf{d}^3) = 1 + E_1 - \frac{E_3}{D_3} - D_1, \quad (99)$$

$$\text{where } E_1 = e_1 + e_2 + e_3, \quad E_2 = e_1e_2 + e_2e_3 + e_3e_1, \quad E_3 = e_1e_2e_3, \quad (100)$$

and  $D_1 = \Sigma_3$ ,  $D_2 = d_1d_2 + d_2d_3 + d_3d_1$ ,  $D_3 = d_1d_2d_3$ . Substituting a correspondence (98) into (100) we calculate,

$$\begin{aligned} E_1 &= 2D_1 + \frac{3}{2}F(\mathbf{d}^3), \quad E_2 = 3\left[D_1 + \frac{1}{2}F(\mathbf{d}^3)\right]^2 - 2D_1\left[D_1 + \frac{1}{2}F(\mathbf{d}^3)\right] + D_2, \\ E_3 &= \left[D_1 + \frac{1}{2}F(\mathbf{d}^3)\right]^3 - D_1\left[D_1 + \frac{1}{2}F(\mathbf{d}^3)\right]^2 + D_2\left[D_1 + \frac{1}{2}F(\mathbf{d}^3)\right] - D_3. \end{aligned} \quad (101)$$

Next, inserting (101) into (99) we get

$$F(\mathbf{d}^3) = -D_1 + \sqrt{D_1^2 + 4(D_3 - D_2)}, \quad G(\mathbf{d}^3) = 1 + \frac{1}{2}F(\mathbf{d}^3). \quad (102)$$

Formulas (102) have been derived independently in [12] by analyzing the Apéry set of pseudosymmetric semigroup  $\mathbf{S}(\mathbf{d}^3)$ .

## 7 Almost Symmetric Semigroups $\mathbf{S}(\mathbf{d}^m)$ of Maximal edim

A study of almost symmetric semigroups  $\mathbf{S}(\mathbf{d}^m)$  with maximal edim is motivated by two reasons. First, there are many known results [8], [10], [4] on generic semigroups  $\mathbf{S}(\mathbf{d}_{MED}^m)$  of maximal edim (MED) that makes it reasonable to apply to them the statements of sections 5 and 6. Next, Proposition 1 in [1], at p. 426, is followed by remark: '*not any almost symmetric MED-semigroup of type  $t$  and Frobenius number  $g$  is of the type described in Propos. 11, as following example shows*',

**Example 3**  $\{d_1, d_2, d_3, d_4\} = \{4, 10, 19, 25\}$  ,  $\beta_1 = 6$ ,  $\beta_2 = 8$ ,  $\beta_3 = 3$

$$\begin{aligned} B_4(4, 10, 19, 25) &= \{64, 73, 79\} , \quad S'(4, 10, 19, 25) = \{6, 15, 21\} , \quad \Delta_{\mathcal{H}}(4, 10, 19, 25) = \{6, 15\} , \\ \Delta_{\mathcal{G}}(4, 10, 19, 25) &= \{1, 2, 3, 5, 7, 9, 11, 13, 17, 21\} , \quad F(4, 10, 19, 25) = 21 , \quad G(4, 10, 19, 25) = 12 , \\ Q(4, 10, 19, 25; z) &= 1 - z^{20} - z^{29} - z^{35} - z^{38} + z^{39} - z^{44} + z^{45} + z^{48} - z^{50} + 2z^{54} + z^{60} + z^{63} - z^{64} + \\ &\quad z^{69} - z^{73} - z^{79} , \quad \ell(6, 7, 8, 10, 11) = \wp(6, 7, 8, 10, 11) = 0 , \quad \sum_4 = 58 . \end{aligned}$$

In Example 3 we have calculated the Hilbert series, the Betti numbers and the other characteristics. Thus, there is a necessity to give the most wide description of almost symmetric MED-semigroups.

Start with known results on the MED-semigroups [8], [10] and [4], Corollary 8 :

$$\begin{aligned} F(\mathbf{d}_{MED}^m) &= d_m - m , \quad G(\mathbf{d}_{MED}^m) = \frac{1}{m} \sum_{k=2}^m d_k - \frac{m-1}{2} , \quad \beta_k(\mathbf{d}_{MED}^m) = k \binom{m}{k+1} , \quad (103) \\ t(\mathbf{d}_{MED}^m) &= m - 1 , \quad \min \Delta_{\mathcal{H}}(\mathbf{d}_{MED}^m) = d_m - d_{m-1} , \quad \max \Delta_{\mathcal{H}}(\mathbf{d}_{MED}^m) = d_{m-1} - m , \\ \#\Delta_{\mathcal{H}}(\mathbf{d}_{MED}^m) &= \frac{2}{m} \sum_{k=2}^m d_k - d_m , \quad \#\Delta_{\mathcal{G}}(\mathbf{d}_{MED}^m) = d_m - \frac{1}{m} \sum_{k=2}^m d_k - \frac{m-1}{2} . \end{aligned}$$

We need one more basic entity for  $S(\mathbf{d}^m)$  which plays a key role and facilitates further discussion: the Apéry set  $\mathbb{AP}(\mathbf{d}^m; d_1)$  of semigroup  $S(\mathbf{d}^m)$  with respect to generator  $d_1$  is defined as follows,

$$\mathbb{AP}(\mathbf{d}^m; d_1) := \{s \in S(\mathbf{d}^m) \mid s - d_1 \notin S(\mathbf{d}^m)\} , \quad \#\mathbb{AP}(\mathbf{d}^m; d_1) = d_1 . \quad (104)$$

The generating function  $AP_1(\mathbf{d}^m; z)$  for the Apéry set  $\mathbb{AP}(\mathbf{d}^m; d_1)$  was given in [4], Formula (4.4),

$$AP_1(\mathbf{d}^m; z) = \sum_{s \in \mathbb{AP}(\mathbf{d}^m; d_1)} z^s = (1 - z^{d_1}) \cdot H(\mathbf{d}^m; z) ,$$

and is related to the numerator  $Q(\mathbf{d}^m; z)$  as follows,

$$Q(\mathbf{d}^m; z) = \prod_{j=2}^m (1 - z^{d_j}) \cdot AP_1(\mathbf{d}^m; z) . \quad (105)$$

As for the MED-semigroups, by [8] we get

$$\mathbb{AP}(\mathbf{d}_{MED}^m; m) = \{0, d_2, \dots, d_m\} , \quad AP_1(\mathbf{d}_{MED}^m; z) = 1 + \sum_{k=2}^m z^{d_k} . \quad (106)$$

Now we arrive at the explicit expression for  $Q(\mathbf{d}_{MED}^m; z)$  which is to our knowledge not discussed in literature. By insertion (106) into (105) we obtain,

$$Q(\mathbf{d}_{MED}^m; z) = 1 + \sum_{k=1}^{m-2} (-1)^k [I_{m,k}(z) + J_{m,k}(z)] + (-1)^{m-1} I_{m,m-1}(z) , \quad \text{where} \quad (107)$$

$$I_{m,k}(z) = \sum_{j_1 > \dots > j_{k-1} \geq 2}^m z^{2d_{j_1} + \overbrace{d_{j_2} + \dots + d_{j_k}}^{k-1 \text{ terms}}} , \quad J_{m,k}(z) = k \sum_{j_1 > \dots > j_k \geq 2}^m z^{\overbrace{d_{j_1} + d_{j_2} + \dots + d_{j_{k+1}}}^{k+1 \text{ terms}}} . \quad (108)$$

The number of contributing monomials into  $I_{m,k}(z)$  and  $J_{m,k}(z)$  read

$$\#I_{m,k}(z) = (m-1) \binom{m-2}{k-1}, \quad \#J_{m,k}(z) = k \binom{m-1}{k+1}.$$

Below we give the polynomials  $I_{m,k}(z)$  and  $J_{m,k}(z)$  for small and large indices  $k$ ,

$$\begin{aligned} I_{m,1}(z) &= \sum_{j_1 \geq 2}^m z^{2d_{j_1}}, \quad I_{m,2}(z) = \sum_{j_1 > j_2 \geq 2}^m z^{2d_{j_1} + d_{j_2}}, \quad I_{m,3}(z) = \sum_{j_1 > j_2 > j_3 \geq 2}^m z^{2d_{j_1} + d_{j_2} + d_{j_3}}, \dots, \\ I_{m,m-2}(z) &= \sum_{j_1 > j_2 > \dots > j_{m-2} \geq 2}^m z^{2d_{j_1} + d_{j_2} + \dots + d_{j_{m-2}}}, \quad I_{m,m-1}(z) = z^{\Sigma_m} \sum_{j \geq 2}^m z^{d_j - m}, \\ J_{m,1}(z) &= \sum_{j_1 > j_2 \geq 2}^m z^{d_{j_1} + d_{j_2}}, \quad J_{m,2}(z) = 2 \sum_{j_1 > j_2 > j_3 \geq 2}^m z^{d_{j_1} + d_{j_2} + d_{j_3}}, \dots, \quad J_{m,m-2}(z) = (m-2) z^{\Sigma_m - m} \end{aligned} \quad (109)$$

In the presentation (107) it is easy to recognize the partial polynomials  $Q_k(\mathbf{d}_{MED}^m; z)$  which are contributing to numerator  $Q(\mathbf{d}_{MED}^m; z)$  in accordance with (4),

$$Q_k(\mathbf{d}_{MED}^m; z) = I_{m,k}(z) + J_{m,k}(z), \quad 1 \leq k \leq m-2, \quad Q_{m-1}(\mathbf{d}_{MED}^m; z) = I_{m,m-1}(z). \quad (110)$$

The number of contributing terms  $z^{C_{j,i}}$  into  $Q_k(\mathbf{d}_{MED}^m; z)$  coincides with  $\beta_k(\mathbf{d}_{MED}^m)$ ,

$$\#Q_k(\mathbf{d}_{MED}^m; z) = \#I_{m,k}(z) + \#J_{m,k}(z) = (m-1) \binom{m-2}{k-1} + k \binom{m-1}{k+1} = k \binom{m}{k+1},$$

in accordance with (103). Prove the main Theorem of this section.

**Theorem 7** *Let a numerical MED-semigroup  $\mathbf{S}(\mathbf{d}_{MED}^m)$  be given. Then it is almost symmetric iff for every element  $d_j$  of generating set  $\mathbf{d}_{MED}^m$  there exists its counterpart  $d_{m-j+1}$  such that*

$$d_j + d_{m-j+1} = m + d_m, \quad 1 \leq j \leq m. \quad (111)$$

**Proof** In accordance with (19) a numerical MED-semigroup  $\mathbf{S}(\mathbf{d}_{MED}^m)$  is almost symmetric iff

$$\Delta_{\mathcal{H}}(\mathbf{d}_{MED}^m) = [\mathbb{B}_{m-1}(\mathbf{d}_{MED}^m) \oplus \{-\Sigma_m\}] \setminus \{F(\mathbf{d}_{MED}^m)\}. \quad (112)$$

According to (109) and (110) a set  $\mathbb{B}_{m-1}(\mathbf{d}_{MED}^m) \oplus \{-\Sigma_m\}$  is composed of degrees of monomials  $z^{d_j - m}$  entering the polynomial  $I_{m,m-1}(z)$ . In other words, by (112) we have

$$\Delta_{\mathcal{H}}(\mathbf{d}_{MED}^m) = \{h_2, h_3, \dots, h_{m-1}\}, \quad h_j = d_j - m. \quad (113)$$

However, by definition (13) of the set  $\Delta_{\mathcal{H}}(\mathbf{d}^m)$  for every element  $h_j \in \Delta_{\mathcal{H}}(\mathbf{d}^m)$  there exists its counterpart  $h_j^* \in \Delta_{\mathcal{H}}(\mathbf{d}^m)$  such that  $h_j + h_j^* = F(\mathbf{d}^m)$ . Substituting the expression (113) for the gaps and the expression (103) for the Frobenius number into the last equality we come to the necessary and efficient conditions in the case of the almost symmetric MED-semigroup,

$$d_j + d_j^* = d_m + m, \quad 2 \leq j \leq m-1. \quad (114)$$



Since the tuple  $\mathbf{d}_{MED}^m$  is arranged in accending order,  $m < d_2 < d_3 < \dots < d_{m-1} < d_m$ , then by (114) a set of counterpartners  $d_j^*$  has to be arranged in descending order,  $d_2^* > d_3^* > \dots > d_{m-1}^*$ . Combining both sequences with opposite growth we can verify that (114) could be satisfied for every generator  $d_j$  iff  $d_j^* = d_{m-j+1}$ . A proof can be given combining a way of contradiction with induction on index  $j$  in (111).

Since a case  $j = 1$  is trivial, we start with  $j = 2$ . According to (103), (112) and (113) we have

$$d_2 - m = \min \Delta_{\mathcal{H}}(\mathbf{d}_{MED}^m) = d_m - d_{m-1} ,$$

that satisfy (111). Let equality (111) holds for all  $1 \leq j \leq q$ . Prove, by way of contradiction, that it holds also for  $j = q + 1$ . Indeed, let for elements  $d_{q+1} \in \mathbf{d}_{MED}^m$  and  $d_{m-q} \in \mathbf{d}_{MED}^m$  there exist counterpartners  $d_{q+1}^* \in \mathbf{d}_{MED}^m$  and  $d_{m-q}^* \in \mathbf{d}_{MED}^m$ , respectively, such that  $d_{m-q} > d_{q+1}^*$ . In accordance with (114) write two equalities

$$d_q + d_{m-q+1} = d_{m-q} + d_{m-q}^* \quad \text{and} \quad d_{q+1} + d_{q+1}^* = d_{m-q} + d_{m-q}^* ,$$

which give rise to following inequalities,

$$d_{m-q}^* - d_q = d_{m-q+1} - d_{m-q} > 0 , \quad d_{q+1} - d_{m-q}^* = d_{m-q} - d_{q+1}^* > 0 . \quad (115)$$

Thus, by (115) we arrive at inequality  $d_q < d_{m-q}^* < d_{q+1}$ . However, the last inequality has not solutions because it presumes existence of generator  $d_{m-q}^* \in \mathbf{d}^m$  between  $d_q$  and  $d_{q+1}$  that contradicts the arrangement of the tuple  $d_1 < \dots < d_q < d_{q+1} < \dots < d_m$ . This finishes proof of Theorem.  $\square$

Proposition 1 comes as Corollary of Theorem 7. Indeed, a tuple of arithmetic sequence

$$t + 1, t + 1 + \frac{g}{t}, t + 1 + 2\frac{g}{t}, \dots, t + 1 + g ,$$

with generic term  $d_j = t + 1 + (j - 1) \cdot g/t$  satisfies (111) :  $d_j + d_{m-j+1} = g + 2(t + 1)$ .

Consider another Corollary which follows by Theorem 7.

**Corollary 3** *Let an almost symmetric MED-semigroup  $\mathbf{S}(\mathbf{d}_{MED}^{2n+1})$  be given. Then an element  $d_{2n+1}$  is an odd integer and a sum  $\sum_{i=1}^{2n+1} d_i$  is divisible by  $2n + 1$ .*

**Proof** Since  $t(\mathbf{d}_{MED}^{2n+1}) = 2n$ , then there exists an index  $j = n + 1$  such that  $d_j = d_{2n+1-j+1}$  and by consequence of (111) the following equality holds,  $2d_{n+1} = 2n + 1 + d_{2n+1}$ . Hence, it follows the 1st part of Corollary:  $d_{2n+1}$  is an odd integer. The 2nd part follows if we denote, in accordance with the 1st part,  $d_{2n+1} = 2w + 1$  and calculate,

$$\sum_{i=1}^{2n+1} d_i = (d_1 + d_{2n+1}) + (d_2 + d_{2n}) + \dots + (d_n + d_{n+2}) + d_{n+1} .$$

According to Theorem 7 and the 1st part of this Corollary we obtain

$$\sum_{i=1}^{2n+1} d_i = n(2n+1+2w+1) + n+1+w = (2n+1)(n+1+w) . \quad \square \quad (116)$$

Explicit formulas for the type  $t(\mathbf{d}_{MED}^m)$  and the Betti numbers  $\beta_k(\mathbf{d}_{MED}^m)$  give another opportunity to specify Theorems 5 and 6 in the case of almost symmetric MED-semigroups. Calculate a sum of the Betti numbers  $\beta_k(\mathbf{d}_{MED}^m)$  and check that it satisfies inequality (9),

$$\sum_{k=0}^{m-1} \beta_k(\mathbf{d}_{MED}^m) = (m-2) \cdot 2^{m-1} + 2 . \quad (117)$$

**Theorem 8** *Let an almost symmetric MED-semigroup  $\mathbf{S}(\mathbf{d}_{MED}^m)$  be given. Then*

$$\rho(\mathbf{d}_{MED}^m) = \ell(\mathbf{d}_{MED}^m) . \quad (118)$$

**Proof** Keeping in mind  $\rho(\mathbf{d}^m) - \ell(\mathbf{d}^m) = \delta(\mathbf{d}^m)$ , we prove Theorem for even and odd edim separately. First, consider an almost symmetric MED-semigroup  $\mathbf{S}(\mathbf{d}_{MED}^{2n})$  and calculate the sums of the Betti numbers  $\beta_k(\mathbf{d}_{MED}^{2n})$  of even and odd indices separately.

Keeping in mind  $\beta_{2n-1}(\mathbf{d}_{MED}^{2n}) = 2n-1$  and making sum of (117) with (8) we obtain,

$$\begin{aligned} \beta_1(\mathbf{d}_{MED}^{2n}) + \beta_3(\mathbf{d}_{MED}^{2n}) + \dots + \beta_{2n-3}(\mathbf{d}_{MED}^{2n}) &= (n-1) \cdot (2^{2n-1} - 2) , \\ \beta_2(\mathbf{d}_{MED}^{2n}) + \beta_4(\mathbf{d}_{MED}^{2n}) + \dots + \beta_{2n-2}(\mathbf{d}_{MED}^{2n}) &= (n-1) \cdot 2^{2n-1} . \end{aligned} \quad (119)$$

Applying now Theorem 5, combine the 1st or the 2nd pairs of equalities in (119) and (83) and get

$$(n-1)2^{2n-1} = 2(n-1)4^{n-1} + \frac{1}{2} \delta(\mathbf{d}_{MED}^{2n}) \quad \longrightarrow \quad \delta(\mathbf{d}_{MED}^{2n}) = 0 . \quad (120)$$

Next, consider an almost symmetric MED-semigroup  $\mathbf{S}(\mathbf{d}_{MED}^{2n+1})$  and make similar calculations,

$$\begin{aligned} \beta_1(\mathbf{d}_{MED}^{2n+1}) + \beta_3(\mathbf{d}_{MED}^{2n+1}) + \dots + \beta_{2n-1}(\mathbf{d}_{MED}^{2n+1}) &= (2n-1) \cdot 2^{2n-1} + 1 , \\ \beta_2(\mathbf{d}_{MED}^{2n+1}) + \beta_4(\mathbf{d}_{MED}^{2n+1}) + \dots + \beta_{2n-2}(\mathbf{d}_{MED}^{2n+1}) &= (2n-1) \cdot 2^{2n-1} - 2n . \end{aligned} \quad (121)$$

Applying Theorem 6, combine the 1st or the 2nd pairs of equalities in (121) and (91) and get,

$$(2n-1) \cdot 2^{2n-1} - 2n = (2n-1)(2^{2n-1} - 1) - 1 + \frac{1}{2} \delta(\mathbf{d}_{MED}^{2n+1}) \quad \longrightarrow \quad \delta(\mathbf{d}_{MED}^{2n+1}) = 0 . \quad (122)$$

Thus, combining (120) and (122) we arrive at (118).  $\square$

We finish this section with very specific almost symmetric MED-semigroup  $\mathbf{S}(\mathbf{d}_{MED}^m)$ , related to Proposition 1, when both cardinalities  $\rho(\mathbf{d}_{MED}^m)$  and  $\ell(\mathbf{d}_{MED}^m)$  are vanishing. It enhances an equality (118) in Theorem 8 in this specific case. First, we start with auxiliary Lemma.

**Lemma 8** *Let an almost symmetric semigroup  $S(\mathbf{d}^{2n+1})$  be given and let the sets  $\mathbb{B}_i(\mathbf{d}^{2n+1})$  and  $\overline{\mathbb{B}}_i(\mathbf{d}^{2n+1})$  be defined in (7) and (30). Define four union sets,*

$$\begin{aligned}\mathbb{B}_o(\mathbf{d}^{2n+1}) &= \cup_{i=1}^n \mathbb{B}_{2i-1}(\mathbf{d}^{2n+1}) , & \mathbb{B}_e(\mathbf{d}^{2n+1}) &= \cup_{i=1}^{n-1} \mathbb{B}_{2i}(\mathbf{d}^{2n+1}) , \\ \overline{\mathbb{B}}_o(\mathbf{d}^{2n+1}) &= \cup_{i=1}^n \overline{\mathbb{B}}_{2i-1}(\mathbf{d}^{2n+1}) , & \overline{\mathbb{B}}_e(\mathbf{d}^{2n+1}) &= \cup_{i=1}^{n-1} \overline{\mathbb{B}}_{2i}(\mathbf{d}^{2n+1}) .\end{aligned}\quad (123)$$

*If a set  $\mathbb{Y} = [\mathbb{B}_o(\mathbf{d}^{2n+1}) \cup \overline{\mathbb{B}}_e(\mathbf{d}^{2n+1})] \cap [\mathbb{B}_e(\mathbf{d}^{2n+1}) \cup \overline{\mathbb{B}}_o(\mathbf{d}^{2n+1})]$  is empty then  $\rho(\mathbf{d}^{2n+1}) = 0$ .*

**Proof** Consider a multiset  $\mathfrak{X}_{12}(\mathbf{d}^{2n+1})$  defined in (89) and write its standard representation

$$\mathfrak{X}_{12}(\mathbf{d}^{2n+1}) = \langle \mathbb{X}_{12}(\mathbf{d}^{2n+1}), \sigma_{\mathbb{X}_{12}(\mathbf{d}^{2n+1})} \rangle , \quad (124)$$

where in view of definitions (36) and (43) of multiset operations  $\vee$  and  $\wedge$  the underlying set  $\mathbb{X}_{12}(\mathbf{d}^{2n+1})$  is given by

$$\mathbb{X}_{12}(\mathbf{d}^{2n+1}) = [\mathbb{B}_o(\mathbf{d}^{2n+1}) \cup \overline{\mathbb{B}}_e(\mathbf{d}^{2n+1})] \cap [\mathbb{B}_e(\mathbf{d}^{2n+1}) \cup \overline{\mathbb{B}}_o(\mathbf{d}^{2n+1})] . \quad (125)$$

By comparison of the sets  $\mathbb{Y}$  and  $\mathbb{X}_{12}(\mathbf{d}^{2n+1})$  we conclude that they coincide,  $\mathbb{Y} \equiv \mathbb{X}_{12}(\mathbf{d}^{2n+1})$ . However, by definition (35) of empty multiset a set equality  $\mathbb{X}_{12}(\mathbf{d}^{2n+1}) = \emptyset$  implies a multiset equality  $\mathfrak{X}_{12}(\mathbf{d}^{2n+1}) = \emptyset$ . Thus,  $\rho(\mathbf{d}^{2n+1}) = 0$  and Lemma is proven.  $\square$

Denote by  $\mathbf{c}$  the MED-tuple of  $\text{edim}=2n+1$  such that its generating set is arranged as an arithmetic sequence and, according to Theorem 7, generates an almost symmetric semigroup,

$$\mathbf{c} = \{2n+1, 2n+1+2a, \dots, 2n+1+4na\} , \quad a \in \mathbb{N} , \quad \gcd(2n+1, a) = 1 . \quad (126)$$

**Corollary 4** *Let an almost symmetric MED-semigroup  $S(\mathbf{c})$  defined in (126) be given. Then*

$$\rho(\mathbf{c}) = \ell(\mathbf{c}) = 0 . \quad (127)$$

**Proof** We analyze a set  $\mathbb{X}_{12}(\mathbf{d}^{2n+1})$  defined in (125) for the case  $\mathbf{d}^{2n+1} = \mathbf{c}$  given in (126). Observe that elements of corresponding sets  $\mathbb{B}_o(\mathbf{c})$ ,  $\overline{\mathbb{B}}_o(\mathbf{c})$  and  $\mathbb{B}_e(\mathbf{c})$ ,  $\overline{\mathbb{B}}_e(\mathbf{c})$  defined in (123) through the partial sets  $\mathbb{B}_k(\mathbf{c})$  and  $\overline{\mathbb{B}}_k(\mathbf{c})$  are coming as degrees  $\xi_{q,k}$  and  $\theta_{q,k}$  of monomials  $z^{\xi_{q,k}} \in I_{2n+1,k}(z)$  and  $z^{\theta_{q,k}} \in J_{2n+1,k}(z)$ , respectively, and as their conjugates  $\overline{\xi}_{q,k}$  and  $\overline{\theta}_{q,k}$ ,

$$\left\{ \begin{array}{c} \overline{\xi}_{q,k} \\ \overline{\theta}_{q,k} \end{array} \right\} = - \left\{ \begin{array}{c} \xi_{q,k} \\ \theta_{q,k} \end{array} \right\} + F(\mathbf{c}) + \sum_{j=1}^{2n+1} d_j , \quad z^{\xi_{q,k}} \in I_{2n+1,k}(z) , \quad z^{\theta_{q,k}} \in J_{2n+1,k}(z) . \quad (128)$$

Indeed, by (110) every partial sets  $\mathbb{B}_k(\mathbf{c})$  and  $\overline{\mathbb{B}}_k(\mathbf{c})$  can be decomposed in other two sets,

$$\mathbb{B}_k(\mathbf{c}) = \mathbb{B}_k^I(\mathbf{c}) \cup \mathbb{B}_k^J(\mathbf{c}) , \quad \overline{\mathbb{B}}_k(\mathbf{c}) = \overline{\mathbb{B}}_k^I(\mathbf{c}) \cup \overline{\mathbb{B}}_k^J(\mathbf{c}) , \quad \text{where} \quad (129)$$

$$\mathbb{B}_k^I(\mathbf{c}) = \cup_{q=1}^{\beta_k} \{\xi_{q,k}\}, \quad \mathbb{B}_k^J(\mathbf{c}) = \cup_{q=1}^{\beta_k} \{\theta_{q,k}\}, \quad \overline{\mathbb{B}}_k^I(\mathbf{c}) = \cup_{q=1}^{\beta_k} \{\bar{\xi}_{q,k}\}, \quad \overline{\mathbb{B}}_k^J(\mathbf{c}) = \cup_{q=1}^{\beta_k} \{\bar{\theta}_{q,k}\}. \quad (130)$$

Consider parity properties of these elements. First, note that according to Corollary 3 and (116) the following sum always takes odd values,

$$F(\mathbf{c}) + \sum_{j=1}^{2n+1} d_j = 4an + (2n+1)(2n(a+1)+1).$$

The last equality together with (128) results in important conclusion:

$$\text{Elements } \xi_{q,k} \text{ and } \bar{\xi}_{q,k} \text{ are of opposite parities as well as elements } \theta_{q,k} \text{ and } \bar{\theta}_{q,k}. \quad (131)$$

Consider the elements  $\xi_{q,k}$  and  $\theta_{q,k}$  in more details. By (108) write them as follows,

$$\xi_{q,k} = 2d_{j_1} + \overbrace{d_{j_2} + \dots + d_{j_k}}^{k-1 \text{ terms}}, \quad \theta_{q,k} = \overbrace{d_{j_1} + d_{j_2} + \dots + d_{j_{k+1}}}^{k+1 \text{ terms}}, \quad (132)$$

and recall that a generic term of the sequence (126) reads  $d_j = 2n+1+2(j-1)a$ . Combining it with (131) and (132) we conclude

$$\begin{aligned} 2 \mid \xi_{q,2k+1}, \quad 2 \mid \theta_{q,2k+1}, \quad \text{and} \quad 2 \nmid \xi_{q,2k}, \quad 2 \nmid \theta_{q,2k}, \\ 2 \nmid \bar{\xi}_{q,2k+1}, \quad 2 \nmid \bar{\theta}_{q,2k+1}, \quad \text{and} \quad 2 \mid \bar{\xi}_{q,2k}, \quad 2 \mid \bar{\theta}_{q,2k}. \end{aligned} \quad (133)$$

Thus, by (130) the sets  $\mathbb{B}_{2k+1}^I(\mathbf{c})$ ,  $\mathbb{B}_{2k+1}^J(\mathbf{c})$  and  $\overline{\mathbb{B}}_{2k}^I(\mathbf{c})$ ,  $\overline{\mathbb{B}}_{2k}^J(\mathbf{c})$  comprise the elements divisible by 2, while the sets  $\mathbb{B}_{2k}^I(\mathbf{c})$ ,  $\mathbb{B}_{2k}^J(\mathbf{c})$  and  $\overline{\mathbb{B}}_{2k+1}^I(\mathbf{c})$ ,  $\overline{\mathbb{B}}_{2k+1}^J(\mathbf{c})$  comprise the elements nondivisible by 2.

Next, based on the last conclusion and equalities (123) and (129) we arrive at parity properties:

$$\text{Sets } \mathbb{B}_o(\mathbf{c}) \text{ and } \overline{\mathbb{B}}_e(\mathbf{c}) \text{ comprise only the elements divisible by 2,} \quad (134)$$

$$\text{Sets } \overline{\mathbb{B}}_o(\mathbf{c}) \text{ and } \mathbb{B}_e(\mathbf{c}) \text{ comprise only the elements nondivisible by 2.}$$

Finally, according to (125) the set  $\mathbb{X}_{12}(\mathbf{c})$  is empty since, by (134), two pairs of sets,  $\mathbb{B}_o(\mathbf{c}) \cup \overline{\mathbb{B}}_e(\mathbf{c})$  and  $\mathbb{B}_e(\mathbf{c}) \cup \overline{\mathbb{B}}_o(\mathbf{c})$ , comprise elements of distinct parities. Therefore, by Lemma 8 this implies  $\rho(\mathbf{c}) = 0$ . However, by Theorem 8 the last equality leads immediately to another equality,  $\ell(\mathbf{c}) = 0$ , that finishes our proof.  $\square$

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